CHAPTER 3

PROPULSION SYSTEMS

THROUGHOUT HISTORY, the methods of propulsion that man has been able to control have allowed him to go faster, farther, and higher. Today he uses rockets to send payloads into space. A space mission needs a launch vehicle or booster which will withstand high acceleration and aerodynamic forces as it lifts the payload from the surface of the earth. To be successful, the mission also requires adequate ground equipment to launch and track the flight vehicle, propulsion and guidance to place it on the desired trajectory, reliable electric power sources, and communication equipment to send data back to earth. Finally, mission success requires trained personnel who follow correct procedures to insure that all subsystems operate properly.

Rocket propulsion is vital in any successful space program. Without it, there would be no payloads in space.

Because the scope of the rocket propulsion field is far too complicated to be treated comprehensively in a single chapter, only a limited number of topics are covered here. This chapter presents the theory of rocket propulsion, rocket propellants, types of chemical rocket engines, and advanced propulsion techniques.

THEORY OF ROCKET PROPULSION

Newton's three laws of motion apply to all rocket-propelled vehicles. They apply to gas jets used for attitude control, to small rockets used for stage separation or for trajectory correction, and to large rockets used to launch a vehicle from the surface of the earth. They also apply to nuclear, electric, and other advanced types of rockets, as well as to chemical rockets. Newton's laws of motion are stated briefly as follows:

1. Bodies in uniform motion, or at rest, remain so unless acted upon by an external unbalanced force.
2. The force required to accelerate a body is proportional to the product of the mass of the body and the acceleration desired.
3. To every action there is an equal and opposite reaction.

These laws may be paraphrased and simplified in relating them to propulsion. For example, the *first law* says, in effect, that the engines must be adequate to overcome the inertia of the launch vehicle. The engines must be able to start the vehicle moving and accelerate it to the desired velocity. Another way of expressing this for a vertical launch is to say that the engines must develop more pounds of thrust than the vehicle weighs. Some space missions require slowing the vehicle or changing its direction. An unbalanced force must be applied to accomplish these tasks. Newton's first law is universal and therefore applies not only to a vehicle at rest on a planet, but also to a vehicle in the so-called "weightless" condition of free flight.

F = ma

Several forces must be considered when the *second law* is applied. For example, the accelerating force is the *net force* acting on the vehicle. This means that if a 100,000-lb vehicle is launched vertically from the earth with a 150,000-lb thrust engine, there is a net force at launch of 50,000 lb—the difference between engine thrust and vehicle weight. Here, the force of gravity is acting opposite to the direction of the thrust of the engine.

Propellants comprise approximately 90% of the vehicle's weight at launch. As the engines run, propellants are expended, decreasing the vehicle weight. Therefore, the net force acting on the vehicle increases, and the vehicle accelerates rapidly.

The acceleration and the resulting velocities attained during powered flight are shown in Figure 1. The acceleration and velocity are low at launch and just after launch due to the small net force acting at that time. But both acceleration and velocity increase rapidly as the propellants are ejected. When the first stage engine is shut off and staging occurs, acceleration drops sharply. When the second stage engine ignites, acceleration and velocity will again increase. As more propellants are ejected by the upper stage rocket engines, there will be rapid increases in acceleration and velocity. When the vehicle reaches the correct velocity (speed and direction) and altitude for the mission, thrust is terminated. Acceleration drops to zero after thrust termination, or burnout, and the vehicle begins free flight. For vehicles with three, four, or more stages, similar changes appear in both the acceleration and velocity each time staging occurs. Staging a vehicle increases the velocity in steps to the high values required for space missions.

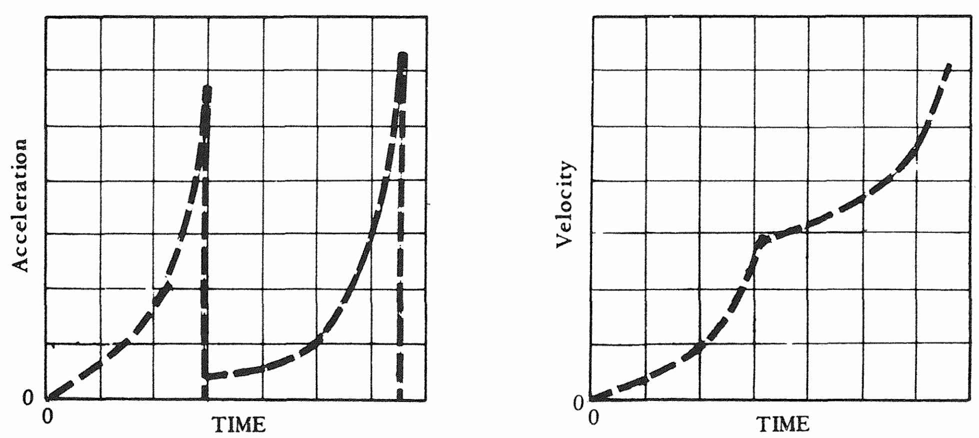


Figure 1. Powered flight of a typical 2-stage rocket.

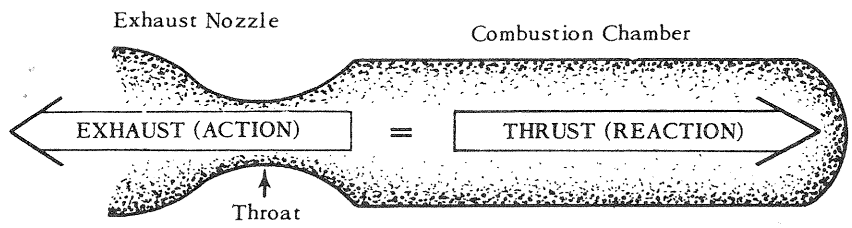


Figure 2. Action-reaction in a rocket motor.

Newton's second law also applies to a vehicle in orbit. Since the vehicle is in a "weightless" condition, the acceleration for a given thrust depends on the vehicle mass, because the vehicle still has mass even though it is "weightless." Remember that mass is the quantity of matter in a body, whereas weight is the force exerted on a given mass by a gravitational field. The mass of a body is the same everywhere, but its weight depends on its mass, the gravitational field, and its position in the gravitational field.

A given thrust will give a large acceleration to a vehicle with a small mass or a small acceleration to a vehicle with a large mass. Even a very small thrust (0.1 lb or less) operating for a long time can accelerate an earth-orbiting vehicle to the velocities needed to go to the other planets of our solar system. The section of this chapter on low thrust engines discusses such propulsion systems.

To relate Newton's *third,* or "action-reaction," *law* to rocket propulsion, consider what happens in the rocket motor (Fig. 2). All rockets develop thrust by expelling particles (mass) at high velocity from their exhaust nozzles. The effect of the ejected exhaust appears as a reaction force, called thrust, acting in a direction opposite to the direction of the exhaust.

Today, practically all exhaust nozzles, whether for a liquid or a solid-propellant rocket, use some form of the de Laval (converging-diverging) nozzle. It accelerates the exhaust products to supersonic velocities by converting some of the thermal energy of the hot gases into kinetic energy.

In the combustion chamber, the burning propellants have negligible unidirectional velocity but have high temperature and pressure. The high pressure forces the gases through the nozzle to the lower pressure outside the rocket. As the gases move through the *converging* section of the nozzle, their temperature and pressure decrease, and their velocity is increased to Mach 1 (the speed of sound) at the smallest cross-sectional area of the nozzle which is called the throat. The gases will attain sonic velocity at the nozzle throat if the combustion chamber pressure is approximately twice the throat pressure.

Since the speed of sound increases with an increase in the temperature of the propagating gas, both sonic velocity and actual linear gas velocity in the throat increase with an increase in gas temperature. Therefore, the higher the gas temperature at the throat, the higher the exhaust velocities that can be generated.

Thrust

Thrust (F) of a rocket is a sum of two terms, "momentum thrust" and "pressure thrust." Momentum thrust is the product of the *propellant rate of flow* and the *velocity of the exhaust* relative to the rocket. Pressure thrust is the product of the maximum cross-sectional *area* of the divergent nozzle section and difference between *exit pressure* of the exhaust and the *ambient pressure* surrounding the rocket. The functional thrust equation below shows these relationships:

$$F = \underbrace{\frac{\dot{W}}{g} v_e}_{\substack{\text{Momentum} \\ \text{Thrust}}} + \underbrace{A_e (P_e - P_o)}_{\substack{\text{Pressure} \\ \text{Thrust}}} \qquad \textbf{(1) (Appendix E)}$$

In the above equation:

F = Thrust developed (lb)
$\dot{W}$ = Weight rate flow of propellants (lb/sec)
g = Acceleration of gravity at the earth's surface (32.2 ft/sec^2)
v_e = Velocity of gases at nozzle exit (ft/sec)
A_e = Cross-sectional area of nozzle exit (in^2)
P_e = Pressure of gases at nozzle exit (lb/in^2)
P_o = Ambient pressure (lb/in^2)

In high thrust rockets, which eject many pounds of propellant (as exhaust products) per second at velocities of several thousands of feet per second, the *momentum thrust is by far the dominant part* of the total thrust. It usually comprises more than 80% of the thrust being developed.

Nozzles and Expansion Ratio

The condition of "optimum expansion" occurs in a rocket nozzle when the pressure of the exhaust gases at the nozzle exit (P_e) is equal to the ambient (atmospheric) pressure (P_o). When $P_e = P_o$, the thrust output of the engine is the maximum that can be obtained at *that* altitude from *that* particular engine. For any given nozzle this condition can occur at only one altitude, i.e., where $P_e = P_o$. Therefore the altitude at which "optimum expansion" occurs depends upon the expansion ratio of the nozzle. "Expansion ratio" is defined as the area of the nozzle exit plane (A_e) divided by the area of the nozzle throat (A_t). It is designated by the letter epsilon. Thus:

$$\epsilon = \frac{A_e}{A_t}. \qquad \textbf{(2)}$$

Figure 3 shows the effect of nozzle expansion ratio on thrust. If the nozzle is cut off to the left of point B, the exit pressure is greater than ambient ($P_e > P_o$), and the nozzle is underexpanded (A_e/A_t is too small). The exhaust gases complete their expansion outside the nozzle.

If the nozzle is extended beyond point B (further increasing the expansion ratio), the exhaust gas exit pressure will be less than ambient ($P_e < P_o$), and the nozzle will be overexpanded. Net thrust will be less because the *pressure thrust loss,* due to the increase in A_e, *is greater than the momentum thrust gained* from the increase in v_e.

Thus, *for a given altitude* (P_o) and a fixed nozzle, maximum thrust is obtained when the nozzle has an expansion ratio so that $P_e = P_o$. As this same engine is flown to a higher altitude, the net thrust output will increase because of an increase in pressure thrust, but net thrust will not be as great as it *could* be if the nozzle expansion ratio could also be adjusted so that $P_e = P_o$ at all altitudes through which the engine operates.

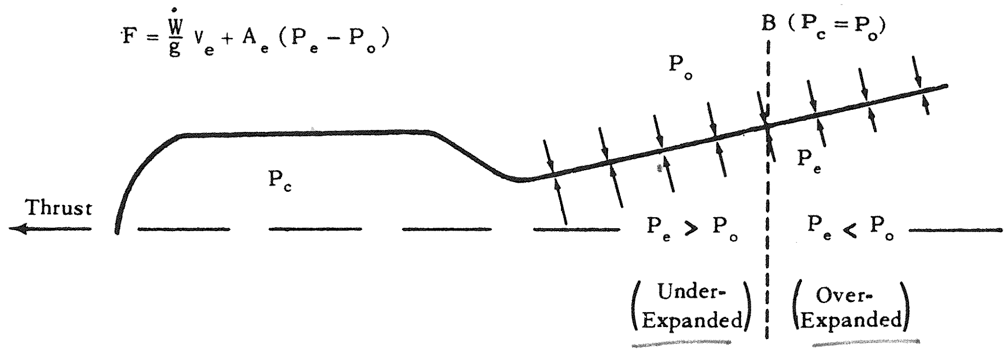


Figure 3. Nozzle expansion.

In summary, the altitude where $P_e = P_o$ is called the design altitude for a specific rocket engine. When $P_e = P_o$, the thrust output of the engine is the maximum that can be obtained at *that* altitude from *that* engine.

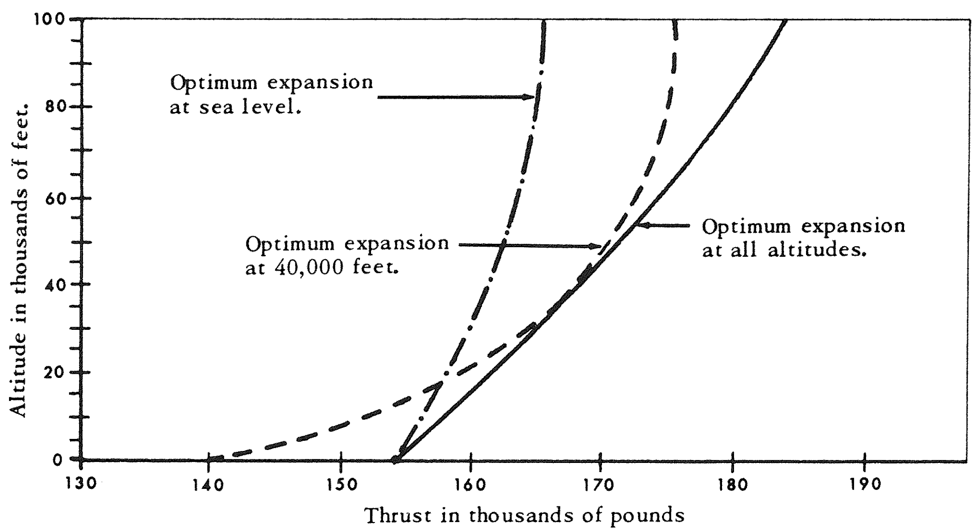


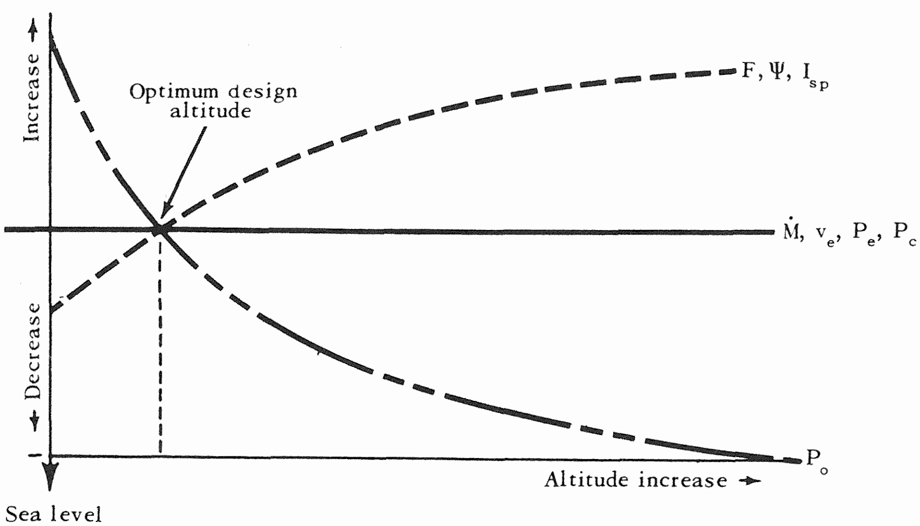
Figure 4. Thrust variation with altitude for nozzles of different expansion ratios.

Nozzles can be designed for optimum expansion at sea level or at any higher altitude. The designer selects the altitude for optimum expansion that gives the best average performance over the powered flight portion of the vehicle trajectory. In multistage rockets the expansion ratios vary in the vehicle stages which operate at different altitudes, with increasingly higher expansion ratios being used for those stages operating at higher altitudes.

Altitude Effects and Thrust Parameters

In evaluating the preceding information, remember that thrust with a given nozzle (fixed area ratio), *regardless* of the design altitude at which optimum expansion occurs, *increases* with altitude until P_o is essentially zero. Large booster engines use nozzles which are overexpanded at launch, achieve optimum expansion at 40,000 to 50,000 feet and have underexpansion above 50,000 feet. In this way they have higher thrust at more altitudes than if they were optimally expanded at launch. (See Figure 4.)

Figure 5 shows which parameters increase as a specific rocket climbs to altitude, which parameters tend to remain constant, and which parameters decrease in value. The rocket will reach an altitude where ambient pressure (P_o) becomes nearly zero. At this altitude, the thrust (F) and the specific impulse (I_{sp}) stabilize at a high value.



Legend

F		Thrust	v_e	=	Velocity of gases at nozzle exit
Ψ	=	Thrust-to-weight ratio	P_e	=	Pressure of gases at nozzle exit
I_{sp}	=	Specific impulse	P_c	=	Combustion chamber pressure
$\dot{M}$	=	Propellant mass rate of flow $= \frac{\dot{W}}{g}$	P_o	=	Ambient atmospheric pressure

Figure 5. A summary of the common rocket engine thrust parameters vs. altitude.

Specific Impulse

Thrust is also important in the definition of specific impulse:

$$I_{sp} = \frac{\text{Thrust (F)}}{\text{Weight rate flow of propellants } (\dot{W})} \qquad (3)$$

Thrust (F) is expressed in pounds, and propellant flow rate ($\dot{W}$) in pounds per second. Thus specific impulse (I_{sp}) is stated in seconds. If the propulsion system has an I_{sp} of 300 seconds, it produces 300 lbs of thrust for every pound of propellant burned per second. I_{sp} is one index of propulsive performance and is related to overall rocket performance.

Increasing I_{sp} improves the propulsion system's ability to increase vehicle velocity. This is the reason that it is frequently quoted to compare the performance of *similar propulsion systems.* I_{sp} should *not* be used alone to compare chemical and electrical or other dissimilar propulsion systems.

In designing a propulsion system, many compromises must be considered because of the many interrelated parameters. However, when a chemical propulsion system is designed only to improve I_{sp}, there are two basic approaches. One is to design a better rocket engine. The other is to use better propellants. These two methods are discussed later in this chapter.

Mass Ratio

Mass ratio is a structural design parameter. It is related to propulsion because the difference between initial (or launch) and final (or empty) vehicle weights is the weight of propellant expended.

$$\text{Mass ratio} = \frac{\text{Initial Weight}}{\text{Final Weight}} = \frac{\text{Weight at Engine Start}}{\text{Weight at Engine Shutdown}} = \frac{W_1}{W_2} \qquad (4)$$

For example, a single-stage, World War II rocket had a mass ratio of 3 to 1 (expressed as $\frac{3}{1}$) and a range of about 200 miles. Mass ratio for the rocket was computed using these figures:

Item	*Launch Weight (lb)*	*Empty Weight (lb)*
Payload	2,000	2,000
Propellants	16,000	0
Other dry weight	6,000	6,000
Total	24,000	8,000

$$\text{Mass ratio} = \frac{\text{Launch Weight}}{\text{Empty Weight}} = \frac{24{,}000}{8{,}000} = \frac{3}{1}$$

It appeared that the way to propel a payload to greater ranges was to build bigger and better rockets. A bigger rocket could carry more propellants, permit longer thrust time, and achieve greater velocities and ranges. In improving the above rocket, the following ICBM design evolved:

Item	*Weight (lb)*	
Payload	200	(note decrease in payload)
Propellants	1,254,000	
Dry weight	29,800	
Launch Weight	1,284,000	

$$\text{Mass Ratio} = \frac{W_1}{W_2} = \frac{(1{,}284{,}000 \text{ lb})}{(30{,}000 \text{ lb})} = \frac{42.8}{1}$$

Obviously, mass ratio had to increase markedly to achieve ICBM ranges. High mass ratios have not been achieved with single stage rockets. A good single stage rocket may have a $\frac{10}{1}$ mass ratio. Mass ratios have been increased by using multistage rockets.

STAGING AND MASS RATIO.—The way that staging increases mass ratio is shown by the following ICBM based on design criteria similar to, but slightly better than, the World War II rocket.

	Item	*Weight (lb)*	
	Third stage		
	Payload	200	= Vehicle weight at engine cut-off (W_2) (1,000 lb)
Stage weight	Dry weight	800	
Stage weight	Propellant	2,500	
	Total	3,500	= Vehicle weight at engine start (W_1)
	Second stage		
	Payload	3,500	= Vehicle weight at engine cut-off (W_2) (10,000 lb)
Stage weight	Dry weight	6,500	
Stage weight	Propellant	25,000	
	Total	35,000	= Vehicle weight at engine start (W_1)
	First stage		
	Payload	35,000	= Vehicle weight at engine cut-off (W_2) (65,000 lb)
Stage weight	Dry weight	30,000	
Stage weight	Propellant	162,500	
	Total	227,500	= Vehicle weight at engine start (W_1)

This rocket carried the 200-lb payload to ICBM ranges just as the scaled-up rocket previously described, but it had a gross weight at launch of only 227,500 lb.—about 17.7% of that of the scaled-up rocket. In the figures above, the initial weights (W_1) and final weights (W_2) for each stage are used to calculate the mass ratio for *each stage operation*. Remember that for each stage the initial weight is the weight of the *vehicle* when the stage's engines are started, and the final weight is the weight of the *vehicle* when that stage's engine is shut-off. There are three mass ratios to be considered:

The third mass ratio is for the final stage only:

$$\text{Third-stage mass ratio} = \frac{W_1}{W_2} = \frac{3{,}500 \text{ lb}}{1{,}000 \text{ lb}} = \frac{3.5}{1}$$

The second mass ratio is for the remainder of the vehicle for second-stage operation:

$$\text{Second-stage mass ratio} = \frac{35{,}000 \text{ lb}}{10{,}000 \text{ lb}} = \frac{3.5}{1}$$

The first mass ratio is for the whole vehicle for first-stage operation:

$$\text{First-stage mass ratio} = \frac{227{,}500 \text{ lb}}{65{,}000 \text{ lb}} = \frac{3.5}{1}$$

OVERALL MASS RATIO.—The overall mass ratio of a multistage rocket is the *product* of the stage mass ratios (See appendix E). Therefore, in the three-stage rocket the overall mass ratio is:

$$\text{Overall mass ratio} = \left(\frac{3.5}{1}\right)\left(\frac{3.5}{1}\right)\left(\frac{3.5}{1}\right) = \frac{42.8}{1}$$

Staging reduces the launch size and weight of the vehicle required for a specific mission and aids in achieving the high velocities necessary for ICBM and space missions. The velocity of the multistage vehicle at the end of powered flight is the sum of velocity increases produced by each of the various stages. The increases are added because the upper stages start with velocities imparted to them by the lower stages.

Thrust-to-Weight Ratio

Comparison of engine thrust to vehicle weight is expressed as a thrust-to-weight ratio (Ψ).

$$\Psi = \frac{\text{Thrust (F)}}{\text{Weight of vehicle (W)}} \qquad (5)$$

A vehicle launched vertically cannot lift off the surface of the earth unless Ψ is greater than 1 ($F > W$). The larger the value of Ψ, the higher the initial vehicle acceleration.

$$a = (\Psi - 1) \text{ g's} \qquad (6) \text{ (Appendix E)}$$

The Ψ of the Minuteman missile is approximately 2, and its initial acceleration is about 1 g, compared with a Titan II missile with a Ψ of about 1.4 and an initial acceleration of about 0.4g.

Mission Velocity Requirements

Figure 6 shows approximate values for the velocity and range relationships of a vehicle launched from the earth. Exact values depend on mission requirements, but a specific velocity is needed at the end of powered flight for each range—that is, *I*ntermediate *R*ange *B*allistic *M*issile (IRBM) range—about 16,000 ft/sec; *I*ntercontinental *B*allistic *M*issile (ICBM)—24,000 ft/sec; and escape velocity near the earth's surface—36,700 ft/sec. Note that the velocity for ICBM range is very close to orbital velocities. In this area, small changes in velocity at the end of powered flight result in large changes in range.

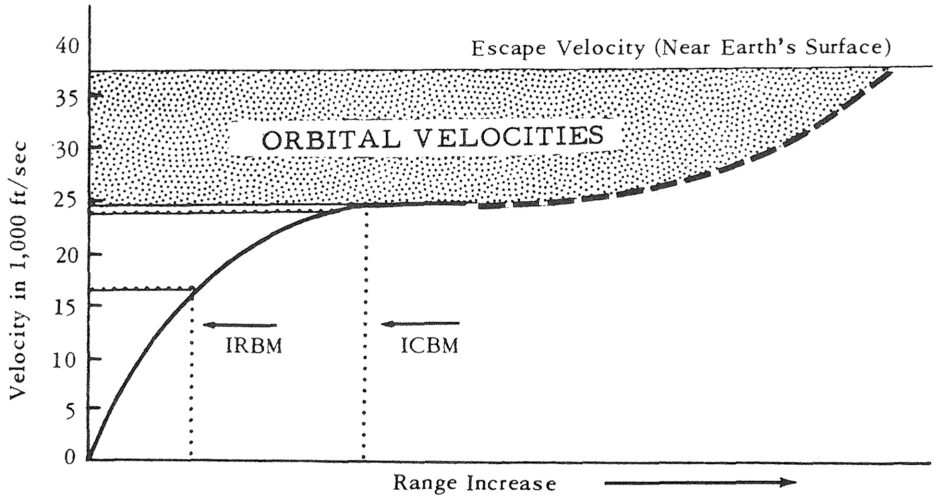


Figure 6. Velocity vs. range of a rocket.

IDEAL VEHICLE VELOCITY CHANGE

Specific impulse and mass ratio directly affect vehicle velocity. The vehicle performance equation shows how they are related to the magnitude of the ideal velocity change (Δv_i) of each stage at burnout or thrust termination.

natural log

$$\Delta v_i = I_{sp}\, g \ln\left(\frac{W_1}{W_2}\right) \qquad (7) \quad \text{(Appendix E)}$$

The ideal Δv is the velocity change the rocket would attain if there were *no gravity, no drag,* and if the *earth did not rotate.*

This simplified equation is derived from Newton's second law and is generally used to find approximate Δv's. The equation will not provide the exact Δv, since atmospheric drag, the earth's rotation, and the changing effects of gravity will cause variations. A precise solution is a computer calculation, but this equation gives a reasonable approximation. Due to losses, the actual Δv's of earth-launched vehicles will be 4,000 to 6,000 ft/sec lower than those computed with this equation. They will also vary with launch latitude and azimuth (see Appendix E).

Equation 7 shows Δv to be directly proportional to two factors: I_{sp} and the natural logarithm (ln) of the mass ratio (MR). In other words, higher Δv's can be achieved by increasing either I_{sp} or MR, or both. The equation does not consider physical size of the rocket or propulsion system. Therefore, in theory, different size rockets with identical I_{sp} and MR would achieve the same ideal Δv. Obviously, larger payloads require larger vehicles and higher thrust engines to achieve the same range or velocity.

The "g" in the equation comes from the conversion of mass to weight at the surface of the earth. Calculations in space use an altitude I_{sp}, which includes the increase in thrust with altitude but measures W at the surface of the earth. Therefore, g is *always* 32.2 ft/sec^2 in propulsion calculations.

Equation 7 applies to all rocket vehicles. For a one-stage rocket the Δv would be the final velocity at thrust termination with the mass ratio based on launch and burnout conditions. In multistage rockets, the ideal Δv is the sum of the Δv's developed by the various stages, with the I_{sp} and MR of each stage used to compute the Δv for that stage's burning time.

The equation may also be used to compute Δv for all space vehicles, including those with a stop and restart capability. When power is applied in flight, a mass ratio combining initial and final weights of the space vehicle for each powered phase is used.

ACTUAL VEHICLE VELOCITY CHANGE

Actual launch vehicle velocity can be expressed as:

$$\Delta v_a = \Delta v_i - \Delta v_l + v_r \qquad (8)$$

In this equation:

Δv_a = Actual Δv

Δv_i = Ideal Δv (from equation 7)

Δv_l = Loss in Δv (gravity and drag) (See Appendix E)

v_r = Variation in Δv due to earth's rotation (See Appendix E)

Two types of problems that may be solved with equation 7 are presented on succeeding pages. A slide rule, math tables, or the graph given in Figure 7 can be used to determine natural logarithms. It should be noted that, since the graph is actually a plot of natural logarithm (1n) values, it may be extended to accommodate values beyond the limits shown. The graph is plotted so that mass ratios are on the vertical axis and the natural logarithms are on the horizontal axis. For example, if $(W_1/W_2) = 8$, this is found on the vertical axis. Then the 1n of 8 = 2.08 is found on the horizontal axis.

Sample Rocket Performance Calculation

A rocket was launched from the earth with an initial weight (W_1) of 296,000 lb. At the end of powered flight, its final weight (W_2) was 40,000 lb. Assuming no earth rotation and no velocity loss due to drag or gravity, calculate the magnitude of the ideal velocity (or Δv_i) at the end of powered flight.

$$I_{sp} = 300 \text{ sec} \quad \text{and} \quad g = 32.2 \text{ ft/sec}^2$$

Use the equation: $\Delta v_i = I_{sp}\, g \ln\left(\frac{W_1}{W_2}\right)$

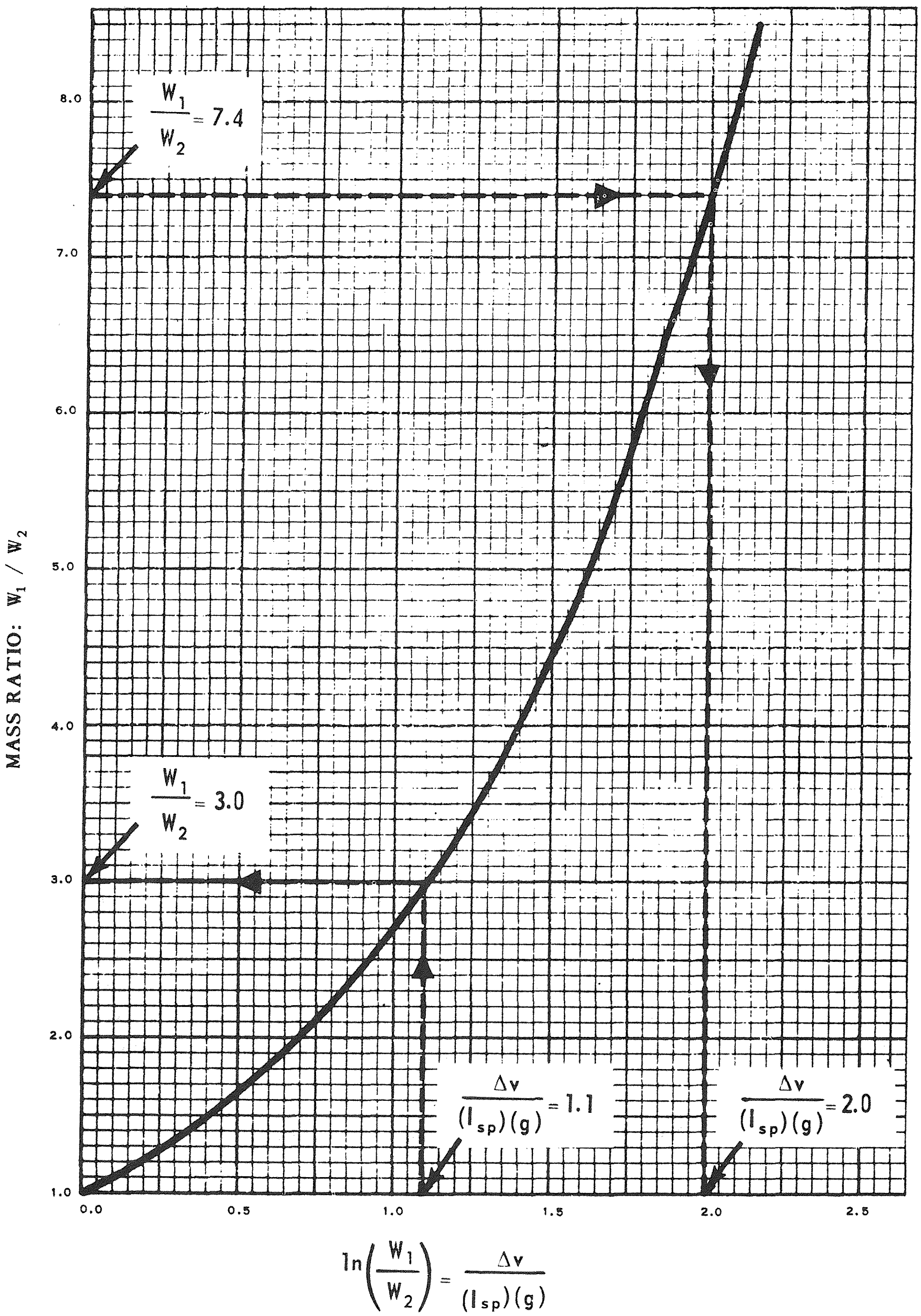


Figure 7. Mass ratio vs. $\Delta v/(I_{sp})(g)$ or $\ln(W_1/W_2)$.

Slide-rule method

a. Substitute

$$\Delta v_i = (300)(32.2)\ \ln \frac{296{,}000}{40{,}000}$$

$$\Delta v_i = (300)(32.2)\ \ln 7.4$$

b. Find: ln 7.4 on slide rule. (above 7.4 on the LL3 scale read 2.0 on the "D" scale.)

$\ln 7.4 = 2.0$

c. Then:

$\Delta v_i = (300)(32.2)(2.0)$

$\Delta v_i = 19{,}320$ ft/sec

Graph method

a. Transpose equation to read:

$$\frac{\Delta v_i}{(I_{sp})(g)} = \ln\left(\frac{W_1}{W_2}\right)$$

b. Calculate value of W_1/W_2

$$\frac{296{,}000}{40{,}000} = 7.4$$

c. Use chart, Figure 7.

d. Enter the graph where $W_1/W_2 = 7.4$ and then

e. Read the value of:

$$\frac{\Delta v_i}{(I_{sp})(g)} = 2.0$$

f. Then, solving for Δv_i

$\Delta v_i = (I_{sp})(g)\ (2.0)$

$\Delta v_i = (300)(32.2)(2.0)$

$\Delta v_i = 19{,}320$ ft/sec

Sample problem for Changing an Orbit
(Finding propellants required)

A space vehicle is coasting in orbit. It has a restart capability and 3,500 lb of propellants on board. A change in orbit is desired. Calculate the propellant required for the orbit change and whether the maneuver can be completed, based on the following data.

$\Delta v_a = 14{,}170$ ft/sec = Magnitude of velocity change required to achieve the new orbit

$I_{sp} = 400$ sec = Specific impulse of the vehicle's engine at *altitude.*

$W_1 = 5{,}000$ lb = Current weight of the vehicle in orbit (measured at surface of the earth)

$g = 32.2\ \frac{ft}{sec^2}$ = Acceleration due to gravity at earth's surface

1. Use the equation: $\Delta v_a = I_{sp}\, g\ \ln (W_1/W_2)$ (There are no drag or gravity losses in space so $\Delta v_i = \Delta v_a$)

2. Transpose to read: $$\frac{\Delta v}{(I_{sp})(g)} = \ln\left(\frac{W_1}{W_2}\right)$$

3. Solve for value of: $\Delta v/(I_{sp})(g)$

$$\frac{\Delta v}{(I_{sp})(g)} = \frac{14{,}170}{(400)(32.2)} = 1.1;\ \text{therefore } 1.1 = \ln(W_1/W_2)$$

4. Complete solution by either of following methods

Slide rule method

a. Since: $\ln(W_1/W_2) = 1.1$ (below 1.1 on "D" scale read 3.0 on the LL3 scale.)

b. Then: $W_1/W_2 = 3.0$

c. Then solving for W_2:

$$W_2 = \frac{W_1}{3.0} = \frac{5{,}000}{3.0} = 1{,}667 \text{ lbs}$$

Graph method

a. Use graph, Figure 7.

b. Enter the graph where: $\Delta v/(I_{sp})(g) = 1.1$ and then Read the value of: W_1/W_2, which is 3.0

c. Then solving for W_2:

$$W_2 = \frac{W_1}{3.0} = \frac{5{,}000}{3.0} = 1{,}667 \text{ lb}$$

d. For both above methods, let W_p = lb of propellants consumed;

e. Then: $W_p = W_1 - W_2 = 5{,}000 - 1{,}667 = 3{,}333$ lb of propellants consumed.

The orbital change can be made because the amount of propellants required is less than the 3,500 lb which are available.

PARKING ORBITS

Most manned missions which proceed to orbit or go farther into space begin by placing a payload into a parking orbit about the earth. The time in the parking orbit is used to wait for appropriate phase angles, for adjustments in launch windows, or for verifying equipment performance before proceeding on the next leg of the mission.

Somewhere between 85 and 100 NM is considered optimum for a parking orbit from a drag and gravity standpoint. The choice of 100 NM allows optimum time in orbit for a manned capsule.

Although the prediction of orbital lifetime is as yet an inexact science, these approximate values illustrate the choices available for parking orbits:

Altitude (NM)	*Expected Time in Orbit (Days)*
85	½
100	3
150	35
200	200
300	4,000

ROCKET PROPELLANTS

Propellants are the working substances that rocket engines use to produce thrust. These substances may be liquid, solid, or gas, but liquids or solids are usually used because they permit more chemical energy to be carried in a particular rocket. Large bulky tanks would be required to contain only a small mass of compressed gas.

This section of the text considers only working substances which are accelerated by the energy released in the chemical combustion (burning) of these substances.

The process involves a fuel and oxidizer reacting chemically to produce high temperature, high pressure gases. Such fuels and oxidizers are called chemical propellants.

Two chemical propellant subject areas are important: theoretical performance characteristics and energy content.

Theoretical Performance of Chemical Propellants

Consider a simplified picture of what happens in a rocket combustion chamber and nozzle (Fig. 8). Burning the propellants releases large amounts of energy and produces high temperature, high pressure gases. The high temperature at point A in the combustion chamber is associated with very rapid motion of the gas molecules. These molecules have high speeds but move in random directions. As they approach the nozzle throat (B), their motion is less random, and they move toward the nozzle exit. At the nozzle exit (C), the largest velocity component of the molecules is parallel to the nozzle axis. The combustion chamber converts the chemical energy of the propellants to high temperature random motion of the gas molecules, and the nozzle orients the velocity or kinetic energy (energy of motion) which gives the rocket its thrust.

As pointed out before, the velocity change of a space vehicle is a function of specific impulse (I_{sp}) and mass ratio. A higher I_{sp} increases the magnitude of the vehicle Δv for a given mass ratio or decreases the mass ratio necessary for a

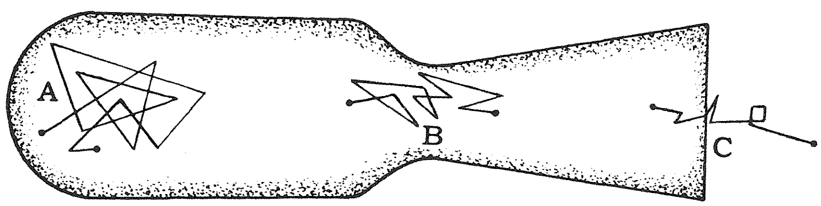


Figure 8. Velocity of gas molecules after combustion in a rocket engine.

given vehicle Δv. Thus, I_{sp} is a measure of how well an engine converts chemical energy into velocity.

Since each mission in space is associated with a required Δv, I_{sp} determines if it is possible to perform a space mission with the mass ratios which are obtainable. For example, a mission to launch from the earth, land on the moon, and return to earth requires a series of Δv's totaling about 59,000 mph. If the engines use propellants with low I_{sp} and can produce Δv's of only 55,000 mph for a particular mass ratio or payload, the space vehicle cannot complete the mission. Engines and propellants with higher I_{sp} are needed for more difficult space missions.

Theoretical Specific Impulse

The theoretical specific impulse of chemical propellants in an ideal rocket is as follows:

$$I_{sp} = 9.797 \sqrt{\left(\frac{k}{k-1}\right)\left(\frac{T_c}{m}\right)\left[1-\left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]} \quad (9) \text{ (Appendix E)}$$

Where:

I_{sp} = Specific impulse (sec)

k = Average ratio of specific heats (C_p/C_v) of the combustion products. C_p is the specific heat of gases at constant pressure. C_v is the specific heat of gases at constant volume.

P_e = Nozzle-exit pressure (psia)

P_c = Combustion-chamber pressure (psia)

m = Average molecular weight of combustion products (lb/mole)

T_c = Combustion chamber temperature (°R). (Degrees Rankine is the absolute temperature and is equal to temperature in degrees F + 459.7°.)

The actual values of C_p, C_v, and k depend on the composition of the gas and its temperature. The combustion products in a rocket chamber and nozzle are a mixture of gases which vary in temperature from about 5,500°R in the combustion chamber to about 3,000°R at the nozzle exit; k is an average value of C_p/C_v for these temperatures (1.2 - 1.33 for chemical engines).

A molecule is the smallest quantity of matter which can exist by itself and retain all the properties of the original substance. For example, a molecule of water is the smallest quantity which has all the properties of water. A molecule of water contains two atoms of hydrogen whose combined atomic weight is about 2, and one atom of oxygen whose atomic weight is 16. Since a molecule is such a small amount of a substance, a mass numerically equal to the combined atomic weights, the molecular weight (called the mole) is used to give the equation a workable mass. In the English system of units a mole of water would be 18 pounds, and is usually called a pound-mole of water. Since the combustion products in a chemical rocket are a mixture of many gases, such as water vapor, carbon monoxide, carbon dioxide, hydrogen, and oxygen, *the average molecular weight of the combustion products* is used in the equation for theoretical specific impulse.

The combustion chamber temperature is the temperature obtained from the reaction of the oxidizer and fuel. This temperature depends on the mixture ratio which is computed as follows:

$$\text{Mixture ratio } (r) = \frac{\text{Weight flow rate of oxidizer } (\dot{W}_o)}{\text{Weight flow rate of fuel } (\dot{W}_f)} \quad (10)$$

There are high and low limits to the possible mixture ratios for an oxidizer and fuel combination. If there is not enough oxidizer, or if there is too much oxidizer, combustion does not take place. A mixture ratio which produces complete combustion of both oxidizer and fuel gives the highest temperature. Too much oxidizer, even within the combustion mixture ratio limits, results in a lower temperature and excess oxidizer in the combustion products. Too little oxidizer results in a lower temperature and unburned fuel in the combustion products. Thus, mixture ratio, combustion chamber temperature, and average molecular weight of the combustion products are all interrelated.

Consider the relative effects of the ratio of specific heats, pressure, average molecular weight of combustion products, and combustion chamber temperature on theoretical I_{sp} for these conditions:

$k = 1.2 \qquad m = 20$ lb/mole $\qquad P_c = 1{,}000$ psia

$T_c = 6{,}000°R. \qquad P_e = 14.7$ psia

$$I_{sp} = 9.797 \sqrt{\left(\frac{1.2}{0.2}\right)\left(\frac{6{,}000}{20}\right)\left[1 - \left(\frac{14.7}{1{,}000}\right)^{\frac{0.2}{1.2}}\right]}$$

$$I_{sp} = 9.797\sqrt{(6)\ (300)\ (0.504)}$$

The ratio T_c/m (300) dominates the equation and:

$$I_{sp} \alpha \sqrt{\frac{T_c}{m}} \qquad (11)$$

This states that the theoretical I_{sp} of chemical propellants is directly proportional to the square root of the ratio of the combustion chamber temperature and the average molecular weight of the combustion products. The actual calculation of theoretical I_{sp} is complicated. The solution requires an extensive computation which is best done with an electronic computer.

The relationship among specific impulse, combustion chamber temperature, and the molecular weight of the combustion products is shown in Figure 9. Note that I_{sp} increases as the value of T_c/m increases.

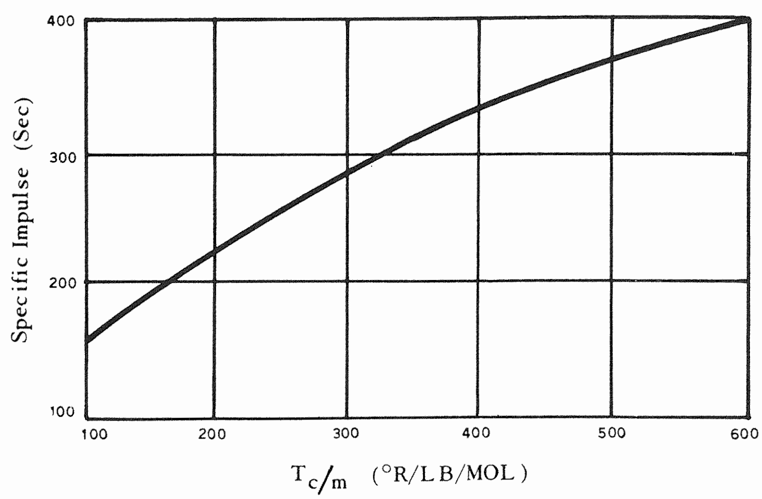


Figure 9. Specific impulse vs. T_c/M.

The highest T_c may not produce the highest I_{sp}, since a corresponding increase in the average molecular weight of the combustion products may result in a smaller value for T_c/m. For example, most rockets using a hydrocarbon fuel like RP-1 (highly refined kerosene) burn a fuel-rich mixture which does not produce the highest T_c. This fuel-rich mixture produces low molecular weight gases like carbon monoxide and results in low average molecular weight combustion products, a high value for T_c/m, and a high I_{sp}. Burning a mixture of liquid oxygen (LOX) and RP-1 which produces the highest T_c results in: high molecular weight gases like carbon dioxide; a lower value for T_c/m; and a lower I_{sp}.

The importance of low molecular weight combustion products effectively limits rocket fuels to chemical compounds of light elements, such as hydrogen, lithium, boron, carbon, and nitrogen. The use of elements heavier than aluminum (atomic weight 27) generally results in lower I_{sp}. Since hydrogen has the lowest molecular weight of all the elements, a propellant with a high hydrogen content is desirable.

In actual practice, the highest performance rocket fuels are relatively rich in hydrogen. Table 1 lists the *theoretical* I_{sp}'s for various typical liquid propellant combinations. These are calculated for the following conditions: a combustion chamber pressure of 1000 psia, an optimum nozzle expansion ratio, an ambient pressure of 14.7 psia and a state of shifting equilibrium. The state of shifting equilibrium assumes a condition of changing chemical composition of gaseous products throughout expansion in the rocket nozzle. This changing chemical composition results in a higher I_{sp} than exists when the chemical composition of the gaseous products is fixed throughout expansion in the nozzle. The latter condition is called frozen equilibrium. The actual composition is somewhere between these two conditions.

Density Impulse

In some instances, a number of different propellant combinations may provide the same I_{sp}. Choosing a combination may then depend upon a parameter called density impulse. Density impulse is the product of specific impulse and specific gravity (SG) of the propellant:

$$I_d = (I_{sp})\ (SG) \tag{12}$$

This parameter is used to relate propulsion system performance to the volume of the tank required to contain the propellants. Given a choice among propellant combinations each having the same I_{sp}, the combination with the highest SG requires smaller propellant tanks.

Total Impulse

Total impulse is directly related to the vehicle's ideal velocity change. Total impulse (I_t) relates the propulsion systems' thrust (F) to the time of operation or burning time (t_b) and is defined as:

$$I_t = (F)\ (t_b) \tag{13}$$

Note that interchanging the values of thrust and operating time does not change the total impulse. For vertical flight the thrust must exceed the weight of the vehicle in order for it to accelerate. But in orbit, either a high or low thrust would accelerate the vehicle to the same final velocity, provided the total impulse were the same. A higher thrust system accomplishes this through higher acceleration but for a shorter period of time than a low thrust system.

Total impulse can also be defined as:

$$I_t = (I_{sp})\ (W_p) \tag{14}$$

From this it is possible to see that this parameter is used to relate propulsion system performance to the allowable weight for propellants. Given a choice among propellant combinations each having the same I_{sp}, the combination having the largest weight would produce the greatest change in velocity.

Characteristics and Performance of Propellants

Rocket engines can operate on common fuels such as gasoline, alcohol, kerosene, asphalt or synthetic rubber, plus a suitable oxidizer. Engine designers consider fuel and oxidizer combinations having the energy release and the physical and handling properties needed for desired performance. Selecting propellants for a given mission requires a complete analysis of: mission; propellant performance, density, storability, toxicity, corrosiveness, availability and cost; size and structural weight of the vehicle; and payload weight.

LIQUID PROPELLANTS.—The term "liquid propellant" refers to any of the liquid working fluids used in a rocket engine. Normally, they are an oxidizer and a fuel but may also include catalysts or additives that improve burning or thrust. Generally, liquid propellants permit longer burning time than solid propellants. In some cases, they permit intermittent operation; that is, combustion can be stopped and started by controlling propellant flow.

Many liquid combinations have been tested; but no combination has all these desirable characteristics:

1. Large availability of raw materials and ease of manufacture.
2. High heat of combustion per unit of propellant mixture. (For high T_c).
3. Low freezing point (Wide range of operation).
4. High density (Smaller tanks).
5. Low toxicity and corrosiveness (Easier handling and storage).
6. Low vapor pressure, good chemical stability (Simplified storage).

Liquid propellants are classified as monopropellants, bipropellants, or tripropellants.

A *monopropellant* contains a fuel and oxidizer combined in one substance. It may be a single chemical compound, such as nitromethane, or a mixture of several chemical compounds, such as hydrogen peroxide and alcohol. The compounds are stable at ordinary temperatures and pressures, but decompose when heated and pressurized, or when the reaction is started by a catalyst. Monopropellant rockets are simple since they need only one propellant tank and the associated equipment.

A *bipropellant* is a combination of fuel and oxidizer which are not mixed until after they have been injected into the combustion chamber. At present, most liquid rockets use bipropellants. In addition to a fuel and oxidizer, a liquid bipropellant may include a catalyst to increase the speed of the reaction, or other additives to improve the physical, handling, or storage properties. Some bipropellants use a fuel and an oxidizer which do not require an external source of ignition but ignite on contact with each other. These propellants are called *hypergolic.*

A *tripropellant* has three compounds. The third compound improves the specific impulse of the basic bipropellant by increasing the ratio T_c/m.

Liquid propellants are also commonly classified as either cryogenic or storable propellants.

A *cryogenic propellant* is one that has a very low boiling point and must be kept very cold. For example, liquid oxygen boils at $-297°$ F, liquid fluorine at $-306°$ F, and liquid hydrogen at $-423°$ F. These propellants are loaded into a rocket as near launch time as possible to reduce losses from vaporization and to minimize problems caused by their low temperatures.

A *storable propellant* is one which is liquid at normal temperatures and pressures and which may be left in a rocket for days, months, or even years. For example, nitrogen tetroxide (N_2O_4) boils at 70° F., unsymmetrical dimethylhydrazine (UDMH) at 146° F., and hydrazine (N_2H_4) at 236° F. However, the term "storable" means storing propellants on earth. It does not consider the problems of storage in space.

One of the cryogenic propellants, LOX, is used with RP–1 in many rocket engines. The H–1 and F–1 engines of the NASA Saturn vehicles use this combination.

Liquid hydrogen (LH_2) and LOX comprise a cryogenic bipropellant. It is used in upper stage engines, such as the RL–10 (Centaur engine), and in the J–2 of the Saturn V.

The 50% UDMH-50% hydrazine fuel (Aerozine 50) with nitrogen tetroxide (N_2O_4) oxidizer is the storable hypergolic bipropellant used in the Titan III. It is classed as a bipropellant since the fuel contains two compounds to improve handling properties rather than to improve I_{sp}.

The fluorine (LF_2) and LH_2 bipropellant with an I_{sp} of 410 seconds (Table 1) shows the improved performance of cryogenics. As a group, they have higher I_{sp} than the storable propellants.

The I_{sp} values in Table 1 represent the maximum *theoretical* values for normal test conditions, which include engine operation at sea level. Actual engines using these propellants at sea level achieve 85 to 92 percent of these values. Engines operating near design altitude frequently achieve specific impulses which exceed these values.

For example, one version of the Atlas engine using RP–1 and LOX is designed for optimum expansion at 100,000 ft. altitude. The I_{sp} of this engine is 215

TABLE 1

Specific Impulse of Liquid Propellant Combinations
(Units are given in seconds)

Oxidizer	FUEL					
	Ammonia	RP-1	UDMH	50% UDMH and 50% hydrazine	Hydrazine (N_2H_4)	Hydrogen*
Liquid oxygen*	294	300	310	312	313	391
Chlorine trifluoride	275	258	280	287	294	318
95% hydrogen peroxide and 5% water	262	273	278	279	282	314
Red fuming nitric acid (15% NO_2)	260	268	276	278	283	326
Nitrogen tetroxide	269	276	285	288	292	341
Fluorine*	357	326	343		363	410

* cryogenic

TABLE 2

Comparison of Payloads for a Three-Stage Launch Vehicle Using Conventional Propellants With One Using High-Energy Propellants in the Upper Stages (Gross weight about one million pounds)

MISSION	PAYLOAD IN TONS	
	Conventional Upper Stages	*High-Energy Upper Stages*
Low earth orbit		
1. Ideal velocity $\approx$ 30,000 ft/sec	15	32
2. Ideal velocity $\approx$ 34,000 ft/sec	9	21
Escape		
3. Ideal velocity $\approx$ 41,000 ft/sec	3	9

sec at sea level and 309 sec at 80,000 ft, compared with the 300 sec listed in Table 1. I_{sp} reported for rocket engines, especially if much higher than the values in Table 1, must be analyzed to determine the altitude and other conditions for which the values are tablulated.

Several of the liquid propellants have theoretical I_{sp} approximately one-third higher than the conventional LOX and RP–1. LOX with LH_2, and LF_2 with LH_2, are examples of bipropellants called *high-energy* propellants. The term "high-energy" evolved from efforts to develop high-performance propellants. All the upper stages of large launch vehicles like the Saturn IB and Saturn V use these propellants.

High-energy propellants in the upper stages of large rockets increase the payload or the mission capability of the vehicle. Consider two three-stage launch vehicles with the same initial gross weight (Table 2). One has high-energy propellants in the upper stages; the other has conventional LOX and RP–1 in the upper stages. The one using high-energy propellants can carry a heavier payload for a given mission, or can perform a more difficult mission with the same payload. A three-stage vehicle with a gross weight of about one million pounds will have the theoretical payloads shown. For mission number 1, the vehicle using conventional propellants has less than half the payload of the one with high-energy propellants. For mission number 3, the conventional vehicle has only one-third the payload of the high-energy vehicle. Also, the conventional vehicle has the same payload (9 tons) for mission number 2 that the high-energy one has for mission number 3. Increasing the number of stages of the vehicle with LOX/RP–1 will increase the payload for the same initial gross weight, but will not approach that of the high energy propellant vehicle.

High-energy fuel tanks are designed differently from those for conventional fuel. Hydrogen tanks are large and bulky because hydrogen has low density and low molecular weight. Therefore, upper stage weight and volume is larger if the upper stage uses hydrogen than if it uses conventional propellants.

Although high-energy propellants in upper stages increase mission capability, these propellants have high reactivity, thermal instability, and low temperatures. Because fluorine is very corrosive and toxic, it is difficult to handle and store. Fluorine, hydrogen, and oxygen are liquids only at very low temperatures. These low temperatures cause many metals to lose their strength, and may cause the freezing of handling equipment such as valves. Fluorine is found in relatively large quantities in nature but is expensive to concentrate in a free state. Because of these and other problems, high-energy propellants are usually used only in upper stages. They are expensive as first stage propellants, and are difficult to store in space vehicles.

SOLID PROPELLANTS.—Solid propellants burn on their exposed surfaces to produce hot gases. Solids contain all the substances needed to sustain combustion. Basically, they consist of either fuel and oxidizer which do not react below some minimum temperature, or of compounds that combine fuel and oxidizer qualities (nitrocellulose or nitroglycerin). These materials are mixed to produce a solid with the desired chemical and physical characteristics.

Solid propellants are commonly divided into two classes: composite (or heterogeneous), and homogeneous.

Composites are heterogeneous mixtures of oxidizer and organic fuel binder. Small particles of oxidizer are dispersed throughout the fuel. The fuel is called a binder because the oxidizer has no mechanical strength. Neither fuel nor oxidizer burns well in the absence of the other. Usually a crystalline, finely ground oxidizer such as ammonium perchlorate is dispersed in an organic fuel such as asphalt; the oxidizer is about 70 to 80 percent of the total propellant weight. There are a large number of propellants of this type.

Homogeneous propellants have oxidizer and the fuel in a single molecule. Most are based on a mixture of nitroglycerine and nitrocellulose, and are called "double-base propellants." The term distinguishes these propellants from many gunpowders which are based on either one or the other of the components. Nitroglycerine is too sensitive to shock and has too much energy to be used safely by itself in an engine. However, it forms a suitable propellant when combined with the less energetic but more stable nitrocellulose. The major components of a typical double-base propellant are:

Component	*Percent of total*
Nitrocellulose	51.38% (propellant)
Nitroglycerine	43.38% (propellant)
Diethyl phthalate	3.09% (plasticizer)
Potassium nitrate	1.45% (flash depressor)
Diphenylamine	0.07% (stabilizer)
Nigrosine dye	0.10% (opacifier)

Note the additives which control physical and chemical properties. Each additive performs a specific function. The *plasticizer* improves the propellant's structural properties. The *flash depressor* cools the exhaust gases before they escape to the atmosphere and promotes smooth burning at low temperatures. The *stabilizer* absorbs the gaseous products of slow decomposition and reduces the tendency of the propellant to absorb moisture during storage. The *opacifier* prevents heat transfer by radiation to sections of the propellant which have not started to burn. (Small flaws in the propellant can absorb enough heat through radiation to ignite the propellant internally, producing enough gas to break it up if an opacifier is not present.)

An ideal solid propellant would possess these characteristics:

1. High release of chemical energy.
2. Low molecular weight combustion products.
3. High density.
4. Readily manufactured from easily obtainable substances by simple processes.
5. Safe and easy to handle.
6. Insensitive to shock and temperature changes and no chemical or physical deterioration while in storage.
7. Ability to ignite and burn uniformly over a wide range of operating temperatures.
8. Nonhygroscopic (non-absorbent of moisture).
9. Smokeless and flashless.

It is improbable that any propellant will have all of these characteristics. Propellants used today possess some of these characteristics at the expense of others, depending upon the application and the desired performance.

The finished propellant is a single mass called a *grain* or stick. A solid propellant rocket has one or more grains which constitute a charge in the same chamber. The use of solid propellants was limited until the development of high-energy propellants, and of the processing techniques for making large grains. Now, single grains are made in sizes up to 22 ft. in diameter.

In addition to being composite or homogeneous, solid propellants are also classed as restricted or unrestricted. They are *restricted* burning charges when inhibitors are used to restrict burning on some surfaces of the propellant. Inhibitors are chemicals which do not burn or which burn very slowly. Controlling the burning area in this manner lengthens the burning time and results in lower thrust. An inhibitor applied to the wall of the combustion chamber reduces heat transfer to the wall and is called a liner.

Charges without an inhibitor are *unrestricted* burning charges. These burn on all exposed surfaces simultaneously. The unrestricted grain delivers a large thrust for a short time, whereas a restricted grain delivers smaller thrust for a longer time. Today, most large solid propellant rockets contain restricted burning charges.

The operating pressure, thrust, and burning time of a solid propellant rocket depend upon: the chemical composition of the propellant; its initial grain temperature; the gas velocity next to the burning surface; and the size, burning surface, and geometrical shape of the grain. A given propellant can be cast into different grain shapes with different burning characteristics.

The thrust of a rocket is proportional to the product of the exhaust velocity and the propellant flow rate. Large thrust requires a large flow rate which is produced by a large burning surface, a fast burning rate, or both. The burning rate of a solid propellant is determined by the speed of the flame passing through it in a direction perpendicular to the burning surface. Burning rate depends on the initial grain temperature, and upon the operating chamber pressure. A solid rocket operates at a higher chamber pressure and thrust when the propellant is hot than when it is cold, but it will burn for a shorter time. The converse is also true. If the chamber pressure is below a minimum value, the propellant will not burn.

Variation in the geometric shape of the grain changes the burning area and thrust output. Figure 10 shows several typical grain shapes.

Basically, there are three ways in which the burning area can change with time. If the area increases as burning progresses, the thrust increases with time, and the grain is called a *progressive* burning grain. If the burning area decreases with time, thrust decreases, and the grain is called a *regressive* burning grain. If the area remains approximately constant during burning, thrust is constant, and the grain is called a *neutral* burning grain. In each case, the type of burning determines how the thrust level changes with time, following initial thrust buildup.

Regressive burning does not produce as much peak acceleration as does neutral or progressive burning, because the thrust decreases as vehicle weight decreases. Typical progressive, neutral, and regressive grains are shown in Figure 11. Lower accelerations are required when rockets are used to propel payloads which cannot withstand high acceleration loading.

The *progressive* grain is a bored cylinder inhibited on the ends and at the chamber wall. All burning occurs on the surface of the central port area. Both the burning area and thrust increase as the propellant burns.

The *neutral* grain is a rod and tube inhibited on the ends and at the chamber wall. All burning occurs on the outside of the rod and on the inside of the tube. As burning progresses, the increasing area on the tube balances the decreasing area on the rod so that total area and thrust remain constant.

The *regressive* grain is a cylinder inhibited on the ends so it can be attached to the chamber. It burns on the outside, and burning area and thrust decrease as burning progresses.

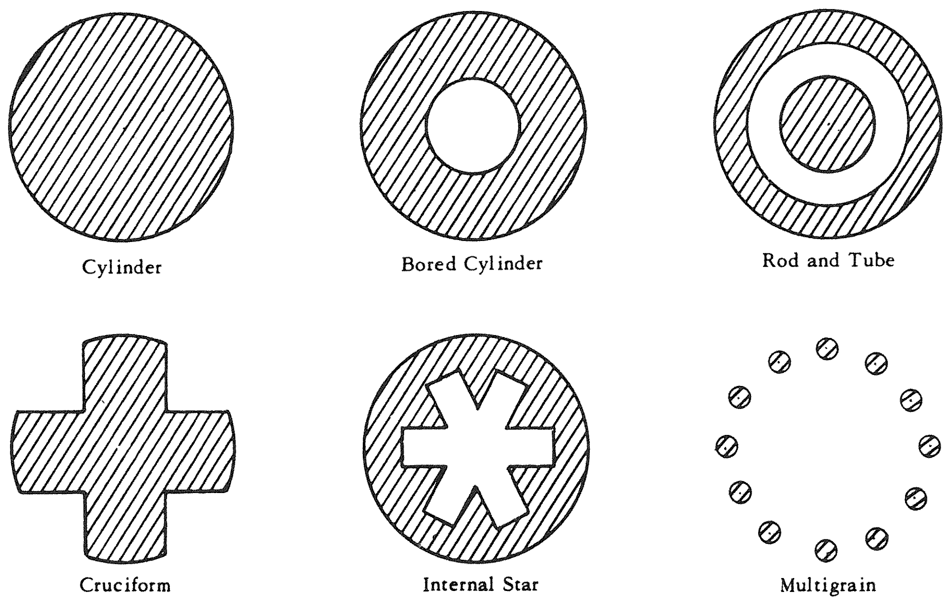


Figure 10. Typical solid propellant grain shapes shown in cross section.

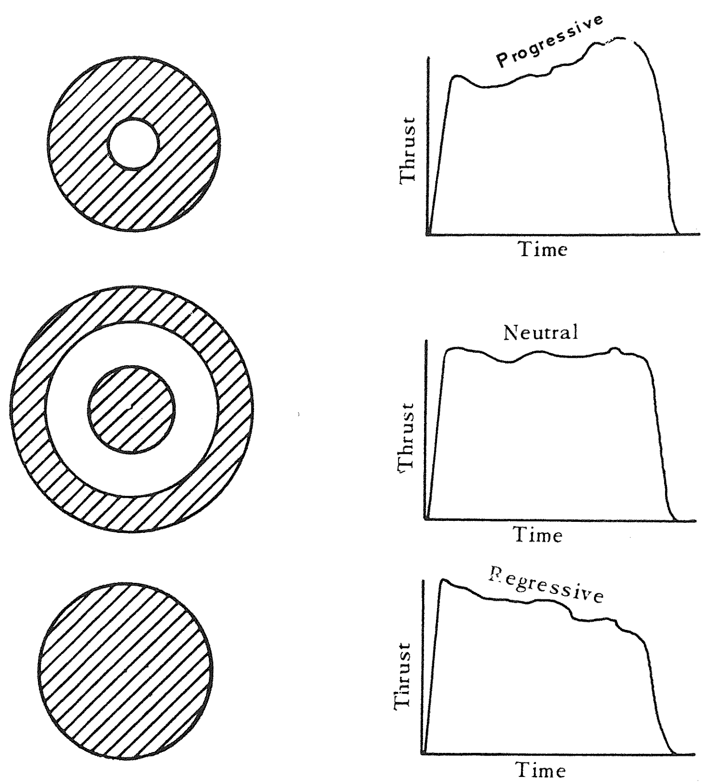


Figure 11. Progressive, neutral, and regressive burning grains.

Theoretical I_{sp} for typical solid propellants is shown in Table 3.

TABLE 3
Specific Impulse of Solid Propellant Combinations

Fuel base	*Oxidizer*	*Specific impulse (seconds)*
Asphalt	Perchlorate	200
Nitrocellulose and nitroglycerine	—	240
Polyurethane	Perchlorate	245
Boron	Perchlorate	270
Metallic hydride	Fluoride	300

Some of the earliest composite propellants had an asphalt and fuel oil base with about 75 percent potassium perchlorate oxidizer. Oil was added to the asphalt to keep it from becoming brittle and cracking at low temperature, but the resultant propellant became soft and began to flow at high temperatures. The specific impulse and operating range of these propellants was rather limited.

The double-base propellants use nitroglycerine and nitrocellulose with the proper additives. These propellants have a higher specific impulse than the asphalt-perchlorate propellants but are more sensitive to shock.

The polyurethane-ammonium perchlorate propellant is a typical example of the specific impulse of present-day synthetic rubber-based composite propellants.

In the past, the specific impulse of composite propellants has been improved by using a higher percentage of oxidizer. This was successful only as long as the propellant retained adequate structural properties, since the fuel-binder gives the propellant its mechanical strength. The specific impulse can also be improved if a light metal, such as aluminum, is added to the fuel. Light metals increase the combustion chamber temperature and lower the molecular weight of the combustion products. The result is a higher value of T_c/m and specific impulse. However, metal in the exhaust gases may cause erosion of the rocket nozzle.

Replacing the inert fuel binder with a high-energy fuel binder containing oxygen (double-base propellants) further increases specific impulse. Some of the propellants used today combine a double-base propellant with a composite propellant and a metal fuel. Propellants under development include boron and metallic hydride fuels in a suitable binder with a perchlorate or a fluoride oxidizer.

As a group, the solid propellants have a lower specific impulse than the liquid propellants, but they have the advantages of simplicity, instant readiness, low cost, potential for higher acceleration, and high density compared to the start-stop capability, high energy, regenerative cooling, and availability of the liquid propellants.

HYBRID PROPELLANTS—Some of the advantages of liquid and solid propellants can be combined in a hybrid rocket. In a hybrid engine, a liquid (usually oxidizer) is stored in one container while a solid (usually fuel) is stored in a second (See Figure 19). The separation of propellants in a hybrid eliminates the dependence of burning time on the grain temperature. The absence of oxidizer in the solid grain also improves its structural properties. The hybrid combines the start-stop advantages of liquid propellants with the high density, instant readiness, and potentially high acceleration of the solid propellants. It is a relatively simple system with high performance. Solid fuel lithium, suspended in a plastic base, burned with a mixture of flourine and oxygen (FLOX) produces a theoretical I_{sp} of about 375 sec.

STORAGE IN SPACE—Propellant storage in space is one of the problems that must be considered in selecting the chemical propellants for space propulsion. The system may use the propellants intermittently over long periods of time, or it may store them for months prior to performing one maneuver. In space, the propellants no longer have the protection of the earth's atmosphere. They must perform in a hostile environment in which they are exposed to a hard vacuum, thermal radiation from the sun, energetic particles such as cosmic rays, and meteoroid bombardments.

Measures are necessary to protect propellants in order to prevent deterioration or loss by evaporation. Liquid propellants must be protected from evaporation and components must be leakproof. Solid propellants must be protected from radiation which affects their burning rate and physical properties. Radiation may also cause changes in liquid propellants. Propulsion systems must withstand temperature extremes when one side faces the sun and the other the coldness of space. They must also be protected from damage caused by micrometeoroids. These are a few of the problems caused by the space environment. Additional research and testing are needed to define all of the effects of space environment on propellants.

CHEMICAL ROCKET ENGINES

One basic method of improving I_{sp} by using more energetic propellants was discussed in detail in the previous section. Designing a better engine for a given propellant is the second basic method. The problem of achieving higher specific impulse generally is not solved by using only one method for improvement. Frequently both methods are used to produce the best results.

This section describes a few typical examples of liquid propellant, solid propellant, and hybrid propellant engines, and suggests a few of the many possible design improvements. The material is limited to the large and comparatively high-thrust rockets, although much of it applies to small rockets, such as those used for vernier and attitude control.

Chemical rocket engines may be classified by several different methods, one of which refers to the type of propellants used. The groups shown in Figure 12 should be remembered when comparisons are made and when methods for improving performance are discussed, because the different groups have different characteristics.

In general, when fast reaction is needed (as in military missiles) ease of handling and simplicity of operation are paramount. In such applications, solid propellant or storable liquid propellant engines are preferred.

For in-space maneuvering where restarts must be made, controllability coupled with high I_{sp}, is desired. The liquid propellant engines have these traits. Of course, compromises have been and will continue to be made based on attainable technology.

The hybrid engines are attempts to combine the advantages of both liquid and solid engines. Several small hybrid engines are in use.

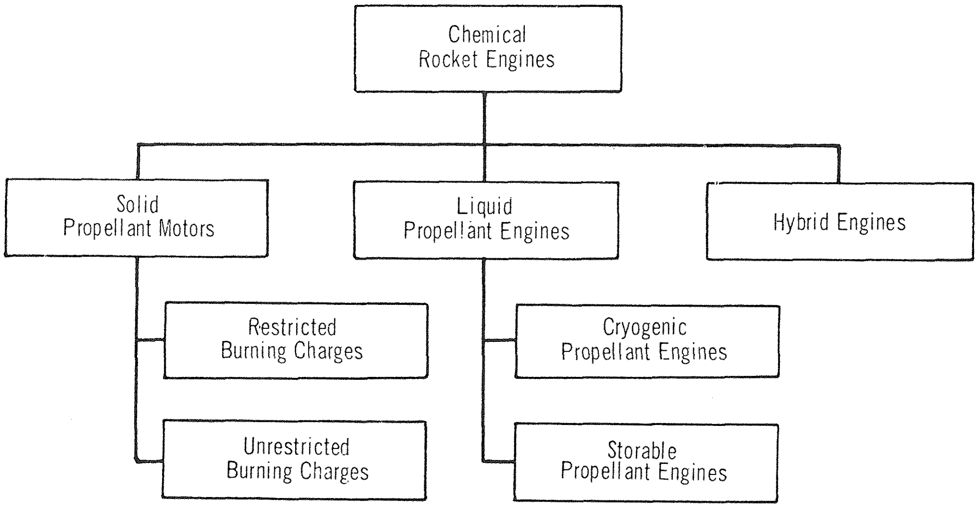


Figure 12. One method of classifying chemical propellant rockets.

Rocket engines are very lightweight for the amount of thrust they produce. Generally, a rocket engine will deliver about 100 pounds of thrust for every pound of engine weight, compared to the better jet engines which produce approximately 10 pounds of thrust per pound of engine weight. If the thrust of a vehicle's engines can be improved without increasing their weight, the propulsion system is improved.

In comparing two or more engines to determine which is best for an application, the entire vehicle and its mission must be evaluated. For example, consider two liquid propellant engines with *similar* thrust levels. The engine with higher I_{sp} would be more efficient and have higher vehicle velocity potential. When engines with similar operating characteristics, propellants, and thrust levels are compared, higher I_{sp} results from better propellant utilization.

A comparison might also be made between two similar liquid engines with similar propellant consumption rates, but different thrust levels. The engine with higher thrust yields higher I_{sp} and is more efficient. Designers try to achieve the highest possible I_{sp} for any given rocket because more I_{sp} yields increased velocity and payload.

Liquid Propellant Engines

A liquid propulsion system consists of propellant tanks, propellant feed system, thrust chamber, and controls such as regulators, valves, and sequencing and sensing equipment. The propellants can be monopropellants, bipropellants, or tripropellants, and may be either storable or cryogenic fluids.

The least complex of these is one designed for monopropellants. Here there is only one propellant tank, a single feed system (usually pressure-fed), and a comparatively simple injector (since mixing of fuel and oxidizer is not required). Monopropellant rockets are in limited use today but do not yet develop high thrust. However, the simplicity of monopropellant engines makes them adaptable and frequently desirable for limited use for attitude control or for small velocity corrections in deep space.

The liquid bipropellant system in common use is more complex. The basic components are shown schematically in Figure 13. Note that two tanks, two feed systems, and multiple injectors are required.

Some bipropellant systems use pressure-fed propellant flow. Here propellants are forced from tanks to engine by pressurizing the tanks with an inactive gas, such as nitrogen or helium. A pressurizing gas can also be created by igniting either a solid propellant grain or some of the vehicle propellants in a gas generator designed for this purpose.

Bipropellants may also be fed to the engine by pumps and gravity, as was done in some earlier booster engines. However, a combination of pumps and pressurizing tanks to provide positive pressure to the pumps feeding the engines is the most commonly used method today. The bipropellant system is complex

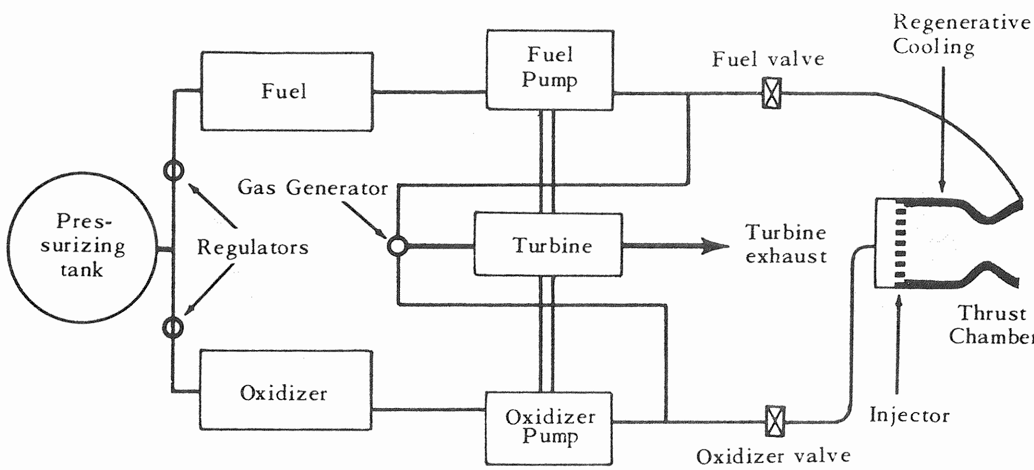


Figure 13. Schematic reproduction of a liquid bi-propellant system.

because of the multiple pumps, the need to maintain the correct oxidizer-fuel mixture, the effect of the injector design upon stable combustion, and the need for thrust chamber cooling.

A centrifugal pump driven by a gas turbine is commonly used to pump the propellants through the injector into the combustion chamber. Gases to drive the turbine are supplied by a separate gas generator, or they are bled from the combustion chamber. The development of reliable turbopumps has presented severe challenges in design, materials, testing, and operational use. In some instances, more than 50% of the design effort for an engine was devoted to the turbopump.

Turbopumps must also pump fuel and oxidizer simultaneously at different rates and be able to withstand high thermal stresses induced by the 1500° F. turbine gases while pumping cryogenic fluids with temperatures as low as −423° F. (liquid hydrogen). Adequate seals must be continuously maintained even at these temperature extremes, since the propellants would explode if they were to come in contact inside the pump. Pump design is also critical because the pumps must develop high propellant pressure. Higher combustion chamber pressure means higher I_{sp}, and pump outlet pressure must be higher than chamber pressure if propellants are to flow into the chamber.

Since the components and controls of the liquid engine can be designed for individual control, the potentials of throttling and multiple restart make these engines attractive for in-space maneuvering.

Solid Propellant Rocket Motors

Solid propellant rocket motors have been in use for thousands of years. History tells of the ancient Chinese using skyrockets for celebrations as well as for weapons. The "rockets' red glare," as used in the National Anthem, indicates the use of rockets during the War of 1812. JATO units, to decrease aircraft take off roll or as take off assist units for lifting heavy loads, are familiar to most Air Force personnel.

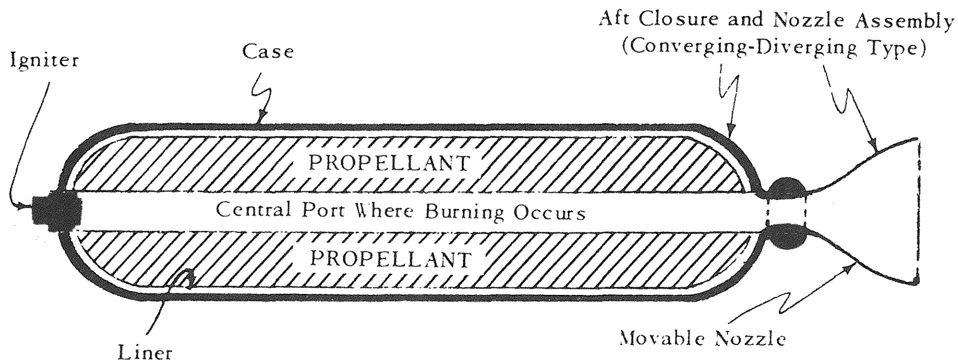


Figure 14. A solid propellant rocket motor.

The solid propellant rocket is comparatively simple. The major components are: the case, which holds the propellants and is the combustion chamber; the igniter; and the converging-diverging nozzle (Fig. 14). Because of its simplicity, the solid motor is inherently more reliable and cheaper to produce than the liquid rocket engine. However, these solids have presented problems, and have a lower I_{sp} than the liquid engines.

As mentioned earlier in the propellant section, the use of additives, new chemicals, and design of high volumetric loading propellant grains is improving the I_{sp} of solids. The other approach for increasing performance is to increase the mass ratio of the motor. Much effort has been expended in this area by designing cases that are lightweight and stronger. Research seems to point to two possible solutions: either make lightweight but strong cases of metals such as titanium, or design filament-wound cases of fiberglass or nylon tape impregnated with epoxy-type glues.

The filament-wound cases can be made even lighter if reinforced propellant grains are designed to assist in supporting the vehicle. Reinforced grains are formed by molding the propellant around aluminum or other metal additive wires. These reinforcing materials are consumed during combustion. Reinforced grain motors are usually regressive burning so that combustion chamber pressure will decrease near the end of burning to allow the use of very lightweight cases.

Because of the relative simplicity of solid motors, they can be made in a variety of sizes from very small attitude control and docking motors with 0.1 lb of thrust to 260 in. diameter motors which produce up to 6 million pounds of thrust.

Thrust Vector Control

Attitude and directional control of a rocket is accomplished by moving the engines or deflecting the exhaust gases. This is called thrust vector control (TVC). The thrust vector for the main and vernier engines is controlled by using flexible mountings or gimbals for the engines. Hydraulic or pneumatic cylinders deflect the engines to change the flight path.

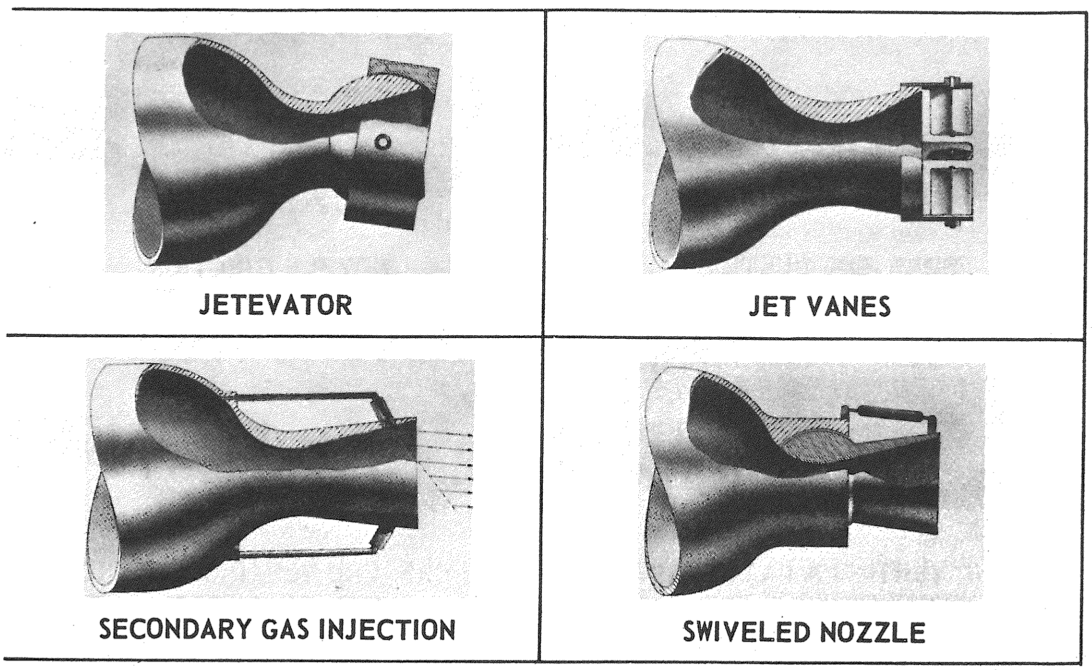


Figure 15. Methods of thrust vector control (TVC).

In solid motors it is impractical to deflect, or swivel, the entire motor because this amounts to moving the entire stage. The thrust vector in solid motors must be controlled at the nozzles. A few of the more common methods employed are: *movable nozzles; controllable vanes* in the nozzles; *jetavators* (slip-ring or collar at the nozzle exit); or *injection of fluid* (hot or cold gases) into nozzles to deflect the exhaust flow and accomplish flight path or attitude changes (see Fig. 15).

Thrust Termination

Thrust termination in liquid rockets is comparatively simple because it is only necessary to stop the flow of propellants. Since this can be done in a reproducible sequence, the residual thrust generated during, and shortly after, engine cut-off will be a known quantity. Liquid engines can be designed so that the thrust level can be changed within limits by varying the rate at which propellants flow into the engine.

In solid motors thrust termination is also simple (Fig. 16). If the combustion chamber pressure is reduced below the critical pressure value for that particular motor, it will, in effect, blow itself out. Blowing of nozzles, blowing out the aft end of the motor, or using thrust termination ports to vent the pressure out the sides of the case are some of the common methods used to reduce the chamber pressure and terminate thrust. However, the thrust termination is not instantaneous. Low levels of thrust may continue for several minutes because of residual burning in the remaining solid grain. The residual burning characteristics of each type of motor must be considered when a very accurate cut-off velocity is needed.

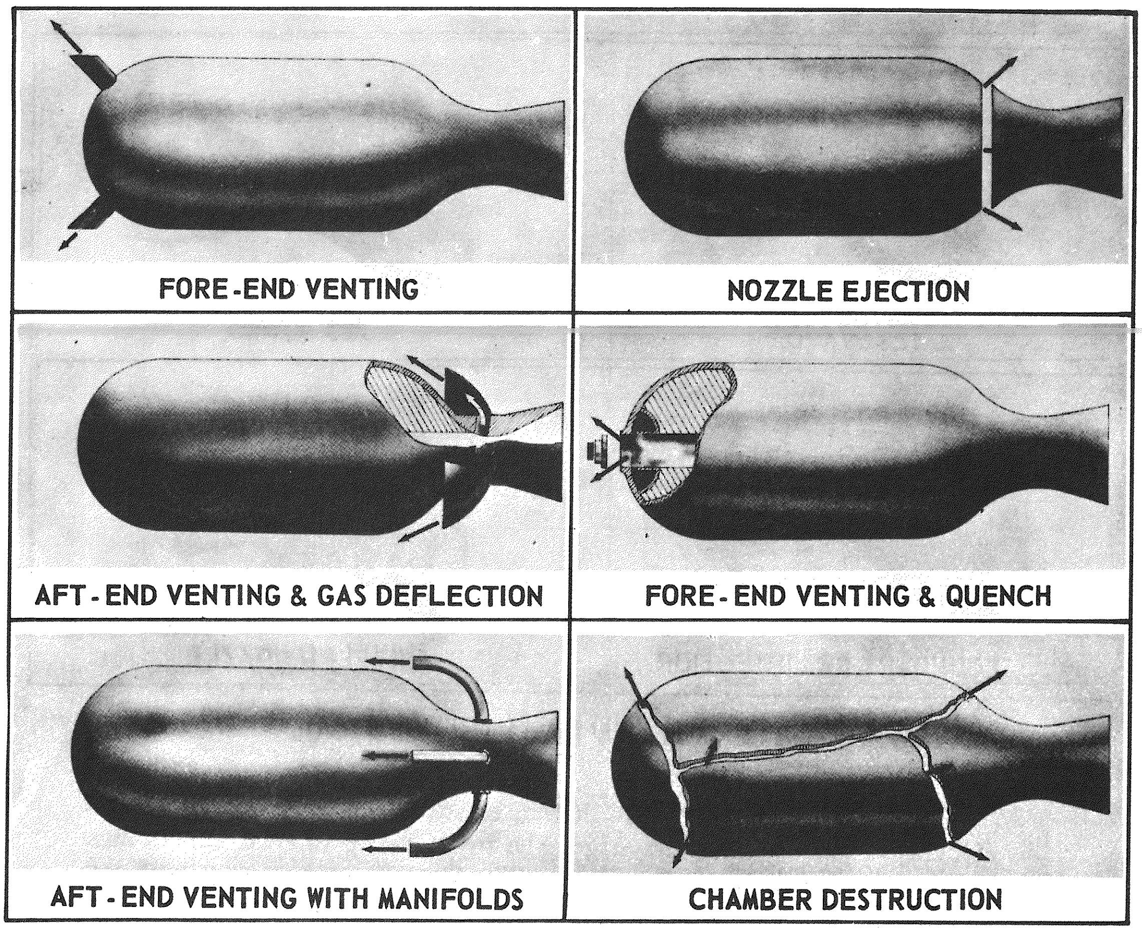


Figure 16. Methods of thrust termination.

Engine Cooling

The combustion chamber of a rocket engine contains gases at temperatures in excess of 5000° F. Large amounts of heat are absorbed by the walls of the combustion chamber and nozzle. Some provision must be made to dissipate heat; otherwise, as the heat is absorbed, the engine wall temperature increases, and the strength of the structural materials decreases. Since the amount of heat absorbed is highest in the nozzle throat, special attention is given to this region.

Most liquid engines are either partially or completely cooled because uncooled liquid rockets are not fired for periods greater than 25 sec. If combustion temperatures were low, it might be possible to use an uncooled engine for longer periods of time; but such is usually not the case in large engines. Cooling methods include regenerative, water, film, sweat (transpiration), radiation, and ablative cooling.

Regenerative cooling circulates fuel or oxidizer through small passageways between the inner and outer walls of the combustion chamber, throat, and nozzle. The heat removed cools the engine and increases the energy of the propellant before it is injected into the combustion chamber. The energy added to the propellant slightly increases the velocity of the exhaust gases and improves engine performance.

Water cooling is regenerative cooling, except that water circulates instead of the fuel or oxidizer. Water cooling is widely used in rocket engine static test stand firings, but not on the flight vehicle.

Film cooling provides a thin fluid film to cover and protect the inner wall of the engine. A protective film is formed on the inner wall by injecting small quantities of the fuel, oxidizer, or a nonreactive fluid at a number of points along the hot surface. The fluid flows along the wall and absorbs heat by evaporation. Film cooling can be used with regenerative cooling for critical parts of the engine where regenerative cooling alone is not sufficient.

Sweat or transpiration cooling uses a porous material for the inner wall of the engine. Coolant passes through this porous wall and is distributed over the hot surface. It is difficult to distribute the coolant uniformly over the surface because the combustion gas pressure decreases between the combustion chamber and the nozzle exit. Manufacturing porous materials so that they are uniform throughout the engine is difficult, and this difficulty limits the use of this method of cooling.

Radiation cooling removes heat from the engine and radiates it to space. Some current liquid engines use a regeneratively cooled nozzle to the 10:1 expansion ratio point and a radiation-cooled extension from that point to the end of the nozzle. This type is lighter than it would be if regenerative cooling were used for the entire nozzle. Current research includes investigating radiation-cooled combustion chambers, nozzles, and nozzle extensions.

Ablative cooling involves coating the surface to be cooled with a layer of plastics and resins. The coating absorbs heat, chars, and then flakes off, carrying heat away from the subsurface being protected.

The cooling methods most commonly used with liquid engines are regenerative cooling, film cooling, or a combination of the two. Since cooling a solid motor would require additional cooling equipment and materials, other protective means have been devised. Inserts of high temperature metal alloys, graphite or ceramics have been used with success. Using multiple nozzles and ablative materials in the nozzle also keeps temperatures within working limits. Heating is also controlled through grain design and use of liners or inhibitors in the case. Combustion gas temperatures in solid motors are generally lower than those in liquid engines.

Nozzles

The development of lighter weight, shorter nozzles has been in progress for several years. There are several reasons for this: (1) to reduce dead weight; (2) to reduce the interstage structure of multistage vehicles; and (3) to optimize the expansion ratio at all altitudes. Several improved nozzles are shown in Figure 17.

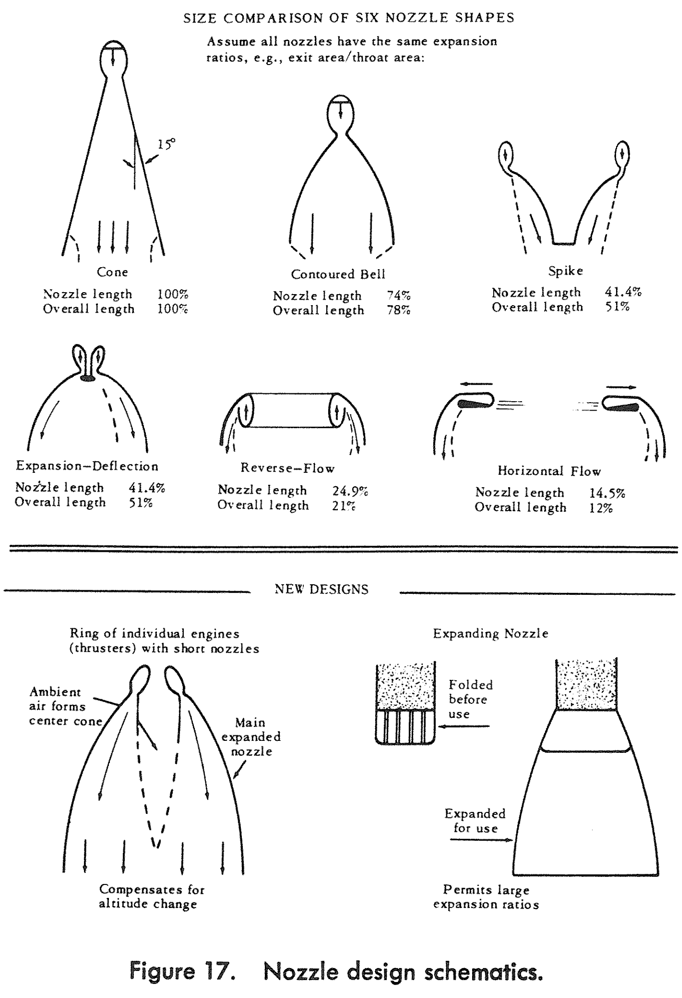


Figure 17. Nozzle design schematics.

Improvements

Several recent concepts have been used to develop bigger and better chemical rockets. Two such developments are the segmented solid and the hybrid rocket.

Solid motors can be segmented or monolithic. The segmented motor is made by stacking separate grains (or segments) together to give the desired thrust level. Using segments greatly eases casting, inspection, and transportation problems. Special methods of assembly insure that cracks do not form between segments and cause the motor to explode.

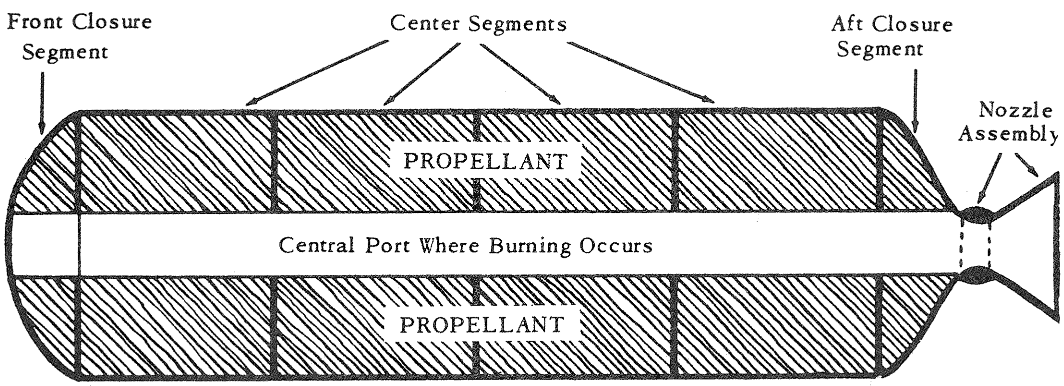


Figure 18. A segmented solid propellant rocket motor.

The segmented solid idea (Fig. 18) has been used to build large diameter motors. These are manufactured in four basic units or "building blocks": the front closure segment, the center segments, the aft closure segment, and the nozzle assembly. The average thrust level of the rocket can be changed by varying the number of center segments, thus controlling the burning area. For example, five-segment, 120 in. diameter, solid motors are used as boosters for the Titan III C with each motor developing 1.2 million lbs of thrust. A seven-segment motor of the same diameter develops 1.6 million lbs of thrust. Above 156 in. diameter, the motors are so large that they will probably be manufactured or cast at the launch site, or transported by water. A monolithic (one piece) motor would be the simplest construction in this case.

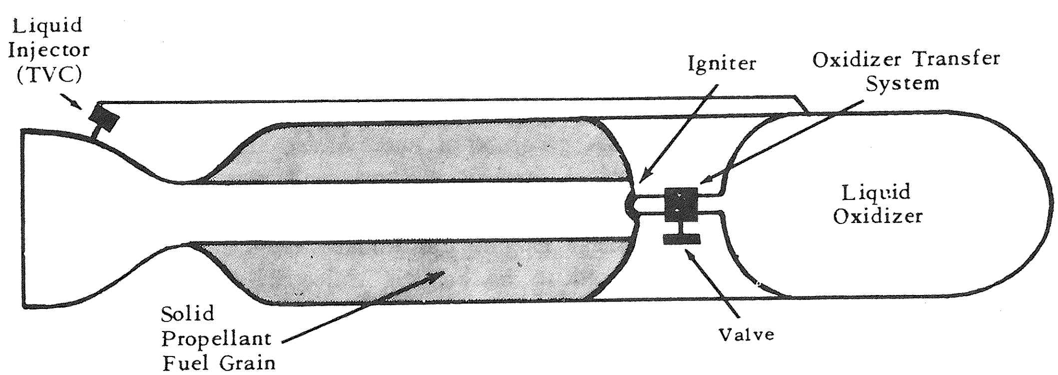


Figure 19. Hybrid rocket engine.

In the hybrid engine (Fig. 19), liquid and solid propellants are used in one engine. The hybrid represents a compromise. It attempts to take advantage of the simplicity and reliability of solids; the higher performance and throttleability of liquids; storability and quick reaction time of solids; and, it reduces the hazard of having both fuel and oxidizer mixed together in one case.

In summary, present trends in liquid rocket engines include very high chamber pressures, high-energy propellants, and simplification of hardware. Solid rockets are being improved by using lighter weight cases, thrust vector control, and uncooled nozzles.

ADVANCED PROPULSION TECHNIQUES

As mission requirements approach or exceed the limits of chemical rockets, new propulsion techniques must be investigated. The new techniques will obey Newton's laws just as contemporary techniques do, but they will use different energy sources and hardware to produce the propulsive force. This force will still be the reaction of the vehicle to mass being ejected at high velocity.

Need for Advanced Designs

The need for advanced designs becomes readily apparent as velocity limitations of chemical rockets are considered. These limitations are shown in Figure 20 for single and multistage vehicles. The graphs include both LOX/RP–1, as well as the higher energy combination LOX/LH_2. *Payload fraction* is the payload weight divided by the gross vehicle weight.

Since there is a practical limit to the number of stages that can be used to increase mass ratio, consider that I_{sp} is proportional to T_c/m. Remember also that T_c and m interact and influence each other in a combustion process. If T_c and m were independent, they could be varied so that the resulting ratio would be higher. The value for m is about 20 in present chemical engines. If m is reduced significantly, a higher performance engine results.

Nuclear Rocket

One program that has followed this approach is the nuclear rocket. It has increased the theoretical I_{sp} to more than 800 sec—double that of current chemical engines. A schematic reproduction of a typical nuclear, solid core, thermal reactor engine is shown in Figure 21. Since liquid hydrogen is used as the propellant, thrust levels are comparable to upper stage chemical engines using hydrogen.

The nuclear rocket was initiated as a joint *A*tomic *E*nergy *C*ommission-USAF project but became an AEC-NASA program of two phases. The early *reactor core design* phase used the Kiwi A and B reactors and was completed in September, 1964. The current phase involves developing the *flight configuration* engine known as NERVA, "*N*uclear *E*ngine for *R*ocket *V*ehicle *A*pplication." Preliminary flight tests are expected to be ballistic trajectory flights with the NERVA functioning as an upper stage main engine. Many areas of study and research are necessary before the nuclear rocket is ready for production. Among these are starting at altitude, and radiation and neutron heating which may

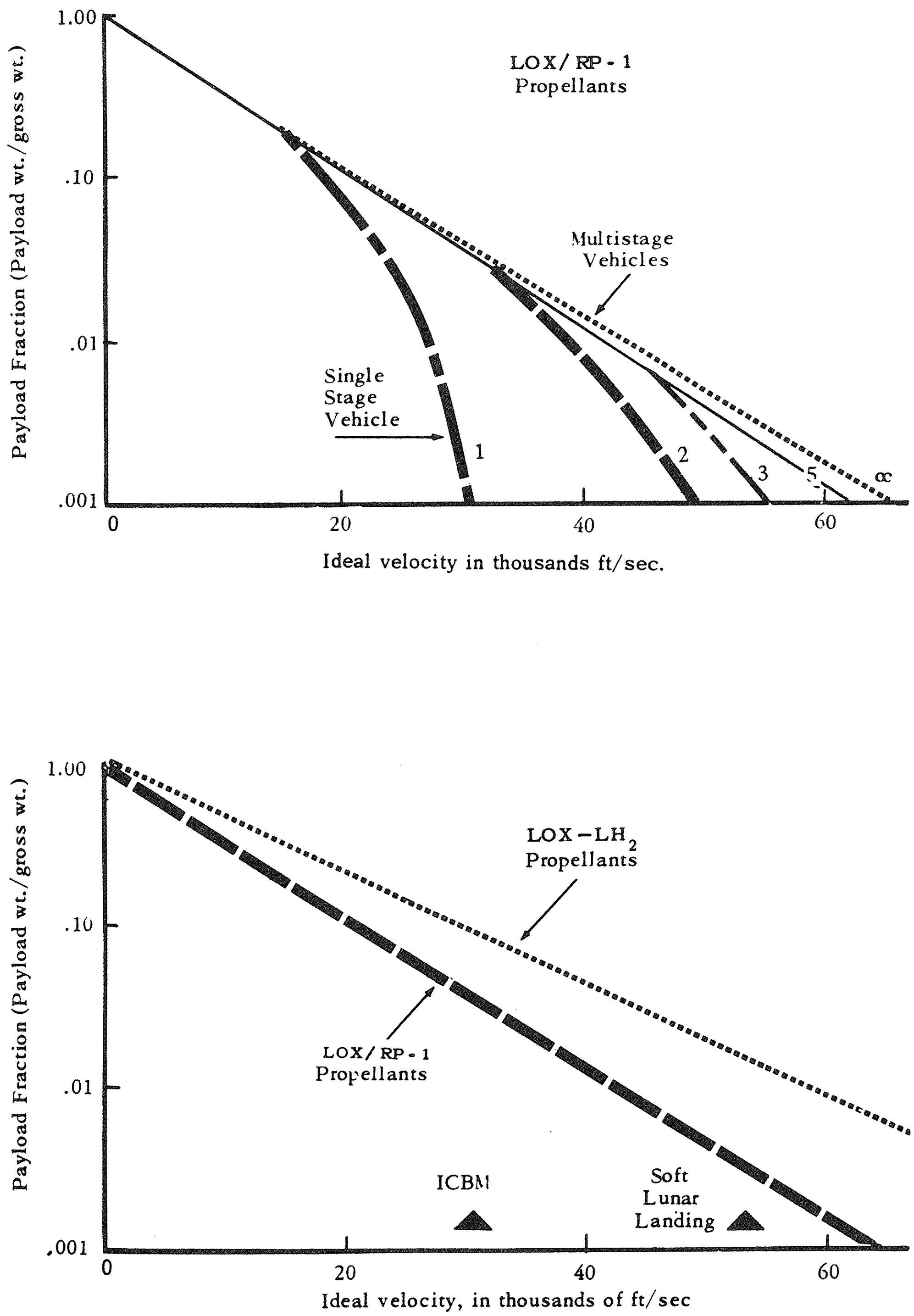


Figure 20. Payload fraction versus ideal velocity.

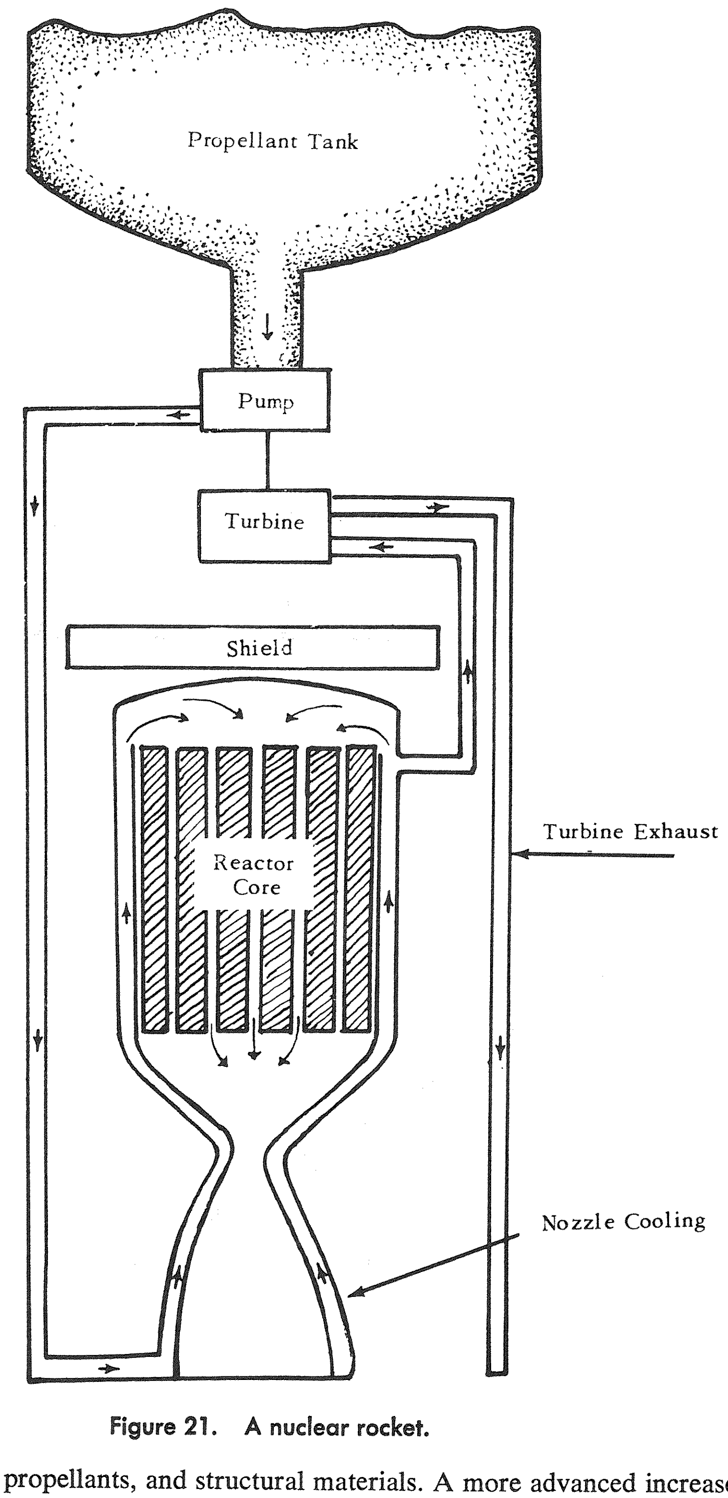


Figure 21. A nuclear rocket.

affect the payload, propellants, and structural materials. A more advanced increased density reactor called PHOEBUS has been tested, and a flyable engine is now being designed.

One theoretical improvement being considered is a high density reactor using fast neutrons. This type of reactor is expected to produce higher performance levels in a smaller package than the thermal (or slow) reactors mentioned above. Another improvement that may prove feasible at some time in the future is a gas core reactor, in which the operating temperature (T_c) could be much higher. This increase in temperature would occur because of the elimination of the solid core or fuel elements used in slow and fast reactors. These structural elements are temperature limited.

The theoretical, comparative performance of thermal reactor engines, fast reactor engines, and chemical engines is shown in figure 22.

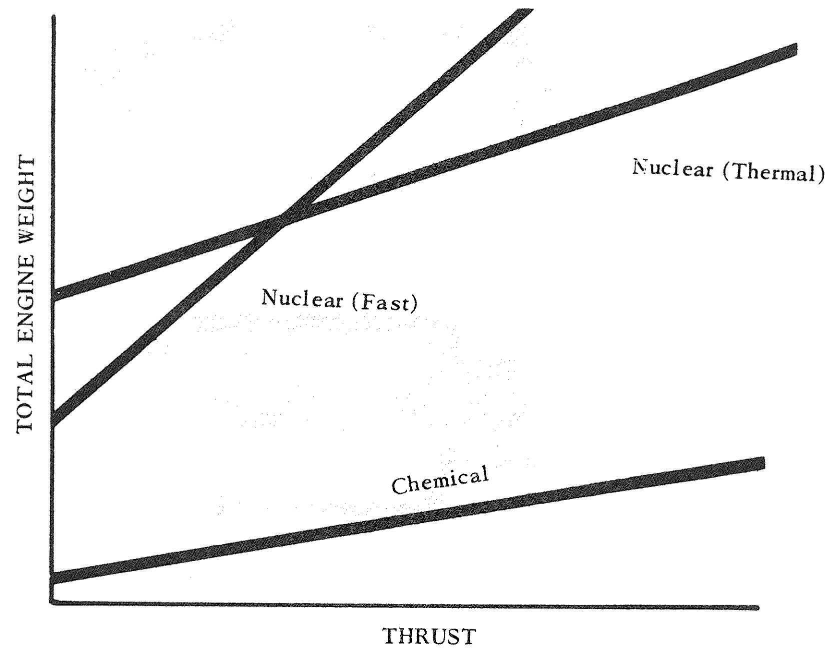


Figure 22. Thrust vs. weight of a rocket.

Low Thrust Rockets

Chemical engines produce high thrust for a short time (minutes), and nuclear reactor engines yield high thrust for hours. Each produces acceleration that stops when the propellant is exhausted.

There are other engines that produce only small amounts of thrust, but they do so for months, or even years. An unbalanced thrust acting for long periods can produce final velocities much higher than those produced by chemical or reactor engines.

The *radioisotope heat cycle* and the *electric* engines produce thrusts measured in micro and millipounds, specific impulses from 700 to 30,000 seconds, and operating times ranging from days to years. These engines cannot lift themselves from the earth, but they can move large payloads through space.

The *radioisotope heat cycle* engines use high energy particle sources such as plutonium and polonium. The particles are stopped by the walls of the isotope container thereby converting their kinetic energy to heat. This heat is used to

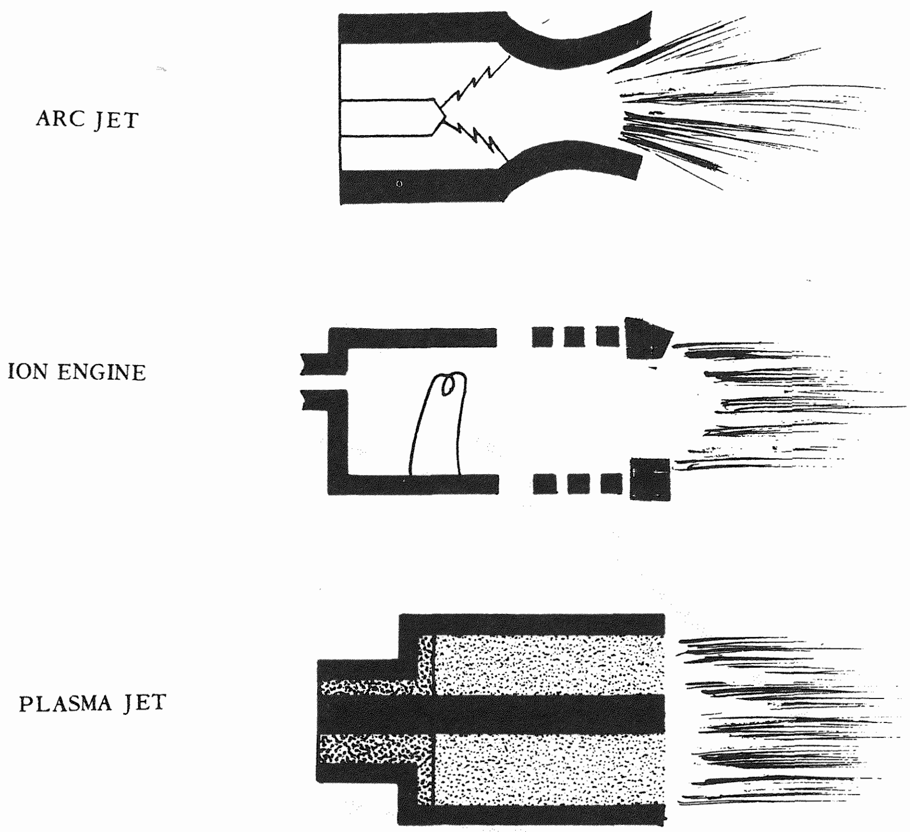


Figure 23. Electric rocket thrust chambers.

raise the temperature and pressure of a propellant which is then expelled through a nozzle.

The DART (*D*ecomposed *A*mmonia *R*adioisotope *T*hruster) is such a device. Here, ammonia is heated by 238 plutonium dioxide to produce 0.1 lb of thrust, and an operating time of at least one year is being sought.

There are basically three types of *electric* engines: arc jet, ion engine, and plasma jet. These are shown schematically in Figure 23.

The *arc jet* uses an electric arc to heat the propellant which is accelerated thermally and ejected as a high velocity plasma from a conventional nozzle.

The *ion engine,* is an electrostatic device which removes electrons from the propellant atoms to form positive ions. These ions are electrostatically accelerated and ejected to produce thrust. Electrons must then be added to make the exhaust electrically neutral to prevent accumulation of a negative charge on the vehicle.

The *plasma jet,* uses electromagnetic force to accelerate and eject the propellant in a plasma form to provide thrust. Electromagnetic force is necessary, since a plasma is a substance which is ionized but is electrically neutral (a plasma consists of an equal number of positive and negative ions, and therefore it has no net charge).

This is, of course, a very simplified presentation. The theory and associated equations for these devices, particularly those for the plasma jet, become quite complex.

The three basic types of electric engines have many subdivisions based on such considerations as design, the method of transferring energy to the propellant, thrust, propellant consumption rates, and I_{sp}. Electric rockets are expected to yield specific impulses of 2,000 to 30,000 sec, or more.

When electric engines are considered, several significant factors must be recognized. All such devices require considerable electrical power, perhaps for years, if they are used for primary propulsion. Only nuclear-electric generators can supply enough power for these long periods of operation. Figure 24 shows a nuclear-electric power and propulsion system. Electric engines have low thrust levels on the order of 0.1 to 0.001 lb. Design improvement will increase these levels. Since electric engines have low thrust-to-weight ratios, they must be used in space where there are weak gravitational fields.

These engines have great potential for some missions. A typical ion rocket uses

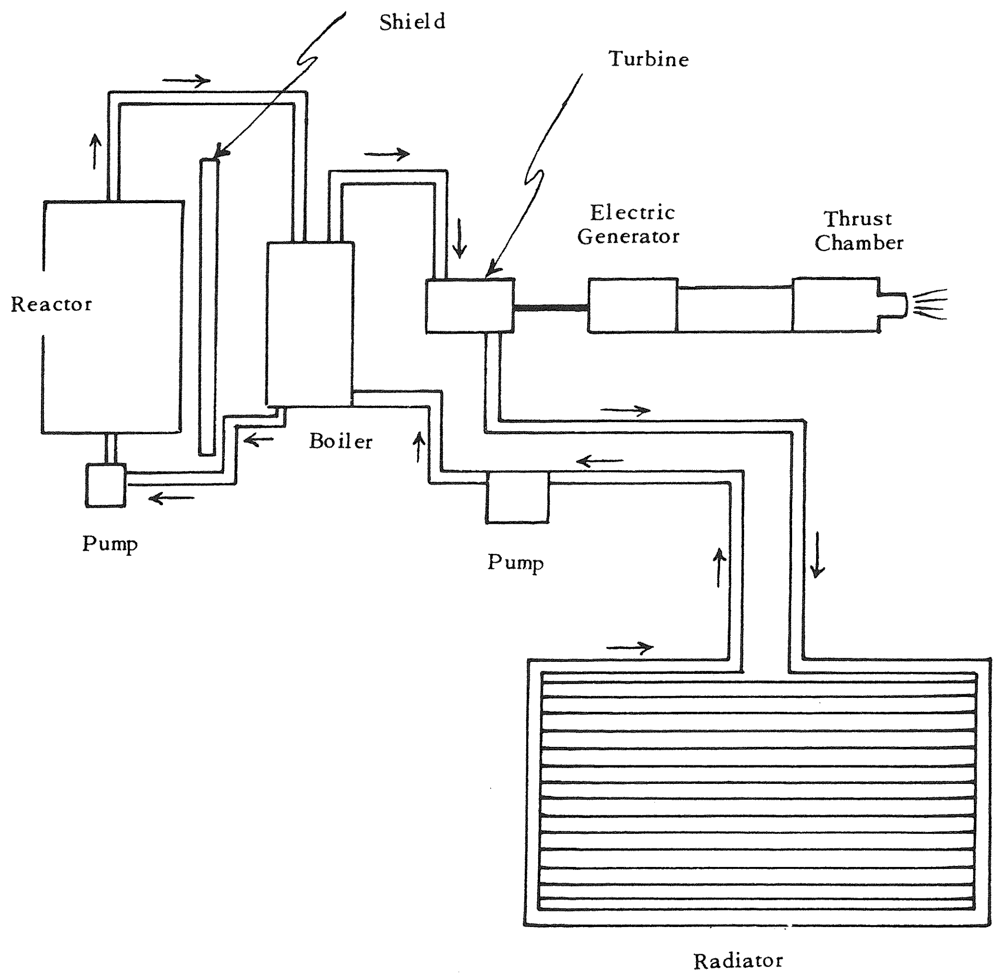


Figure 24. Nuclear-electric power and propulsion system.

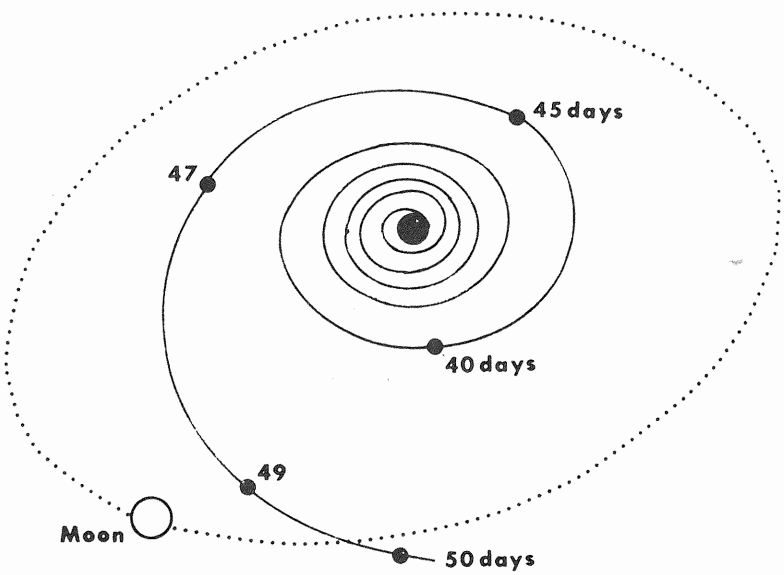


Figure 25. Escape trajectory.

some working fluid (such as cesium) which is vaporized and routed to the "engine." The fluid has electrons stripped from its atoms by passing vaporized cesium over a heated platinum grid. This stripping action leaves positively charged atoms or ions which are accelerated by an electrostatic field and ejected to produce thrust. The stripped-off electrons (negative ions) are also accelerated and introduced into the positive ion stream producing an electrically neutral exhaust. If a difference in potential of about 30,000 volts exists between the platinum grid and the cathode accelerator, ion velocities in the exhaust exceed 500,000 ft/sec.

An electrically powered vehicle with a 2/1 mass ratio and a 20,000 sec specific impulse can be useful in space. Starting from a 200 NM earth orbit, this vehicle might require 2½ to 3 years to reach its peak velocity, but that velocity would be 80 to 100 miles per sec. Compare this with the 5 to 7 mile per sec of our current earth-orbiting satellites.

From the standpoint of a practical mission, the electrically powered vehicle can move a sizable payload to the vicinity of the moon in approximately 50 days, following the trajectory shown in Figure 25. If long flight times can be tolerated, electric engines can move large payloads out to Mars and Venus. If long flight times are not feasible, either chemical or nuclear reactor engines must be used.

For missions beyond Mars and Venus the electric engines come into their own and may easily excel chemical and nuclear reactor rockets, both of which achieve their peak velocities in a rather short powered flight phase. On the other hand, the electric systems continue to accelerate for days, even weeks, and attain much higher final velocities. This means there is a mission crossover point in deep space (in the vicinity of Mars and beyond) at which the electric propulsion

TABLE 4
Mission Summary

TYPE	*MISSIONS*
CHEMICAL PROPULSION	Manned missions near Earth and Moon return. Instrumented probes to Venus and Mars.
NUCLEAR PROPULSION	Heavy payload manned missions in 1970's to Moon, Venus, and Mars and return.
ELECTRIC PROPULSION	Potential for very heavy payloads from Earth orbit to vicinity of Mercury, Jupiter and Saturn (low gravitational field applications only).

systems have the advantages of both shorter flight times and larger payloads. This is shown in Table 4.

Future Propulsion Concepts

Discussion in this chapter has thus far included propulsion systems in production (chemical) and systems in active research and development programs (nuclear and electric). Now concepts for possible future propulsion will be considered.

ADVANCED NUCLEAR—An advanced nuclear concept is one in which propulsive power is derived from a series of controlled nuclear explosions. The force of the explosions impinges on the pusher plate which, in turn, transfers the energy to the main vehicle through a shock absorber system. The shock absorber smooths out the impulses and limits the "G" loading on the main vehicle to an acceptable level. This concept, shown in Figure 26, would provide a controllable specific impulse, and could propel a large payload out to Mars, Venus, or beyond. Payload fractions as high as 45% are possible with this system.

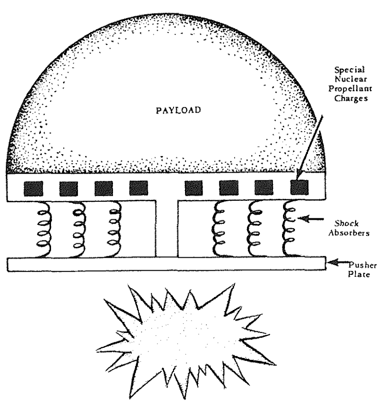


Figure 26. Advanced nuclear impulse concept.

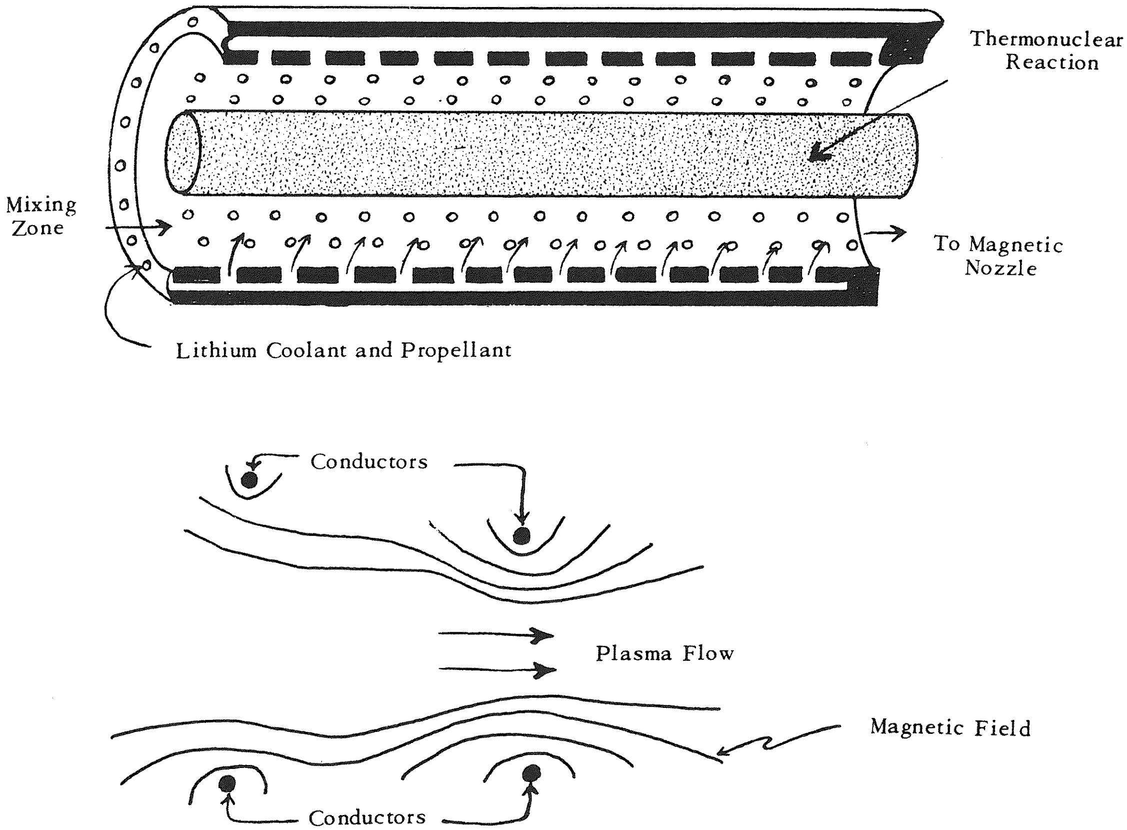


Figure 27. Fusion propulsion concept.

FUSION PROPULSION—Another advanced concept would use a fusion reaction contained by an electromagnetic field (Fig. 27).
The field also heats and controls the reaction, and controls the ejection of mass from the exhaust nozzle. The propellants are two isotopes of hydrogen. Deuterium (H_2) and tritium (H_3) are proposed because they can sustain a fusion reaction when sufficiently heated. This concept may yield a theoretical specific impulse of one million sec.

Scientists have proposed using a porous envelope through which lithium coolant (and propellant) would be forced into the reaction. The lithium would enter into the reaction and be expelled with the other exhaust products. When this is done, more mass flows, lowering the specific impulse to a value of about 4,000 sec, *but* the thrust level rises to a very high value.

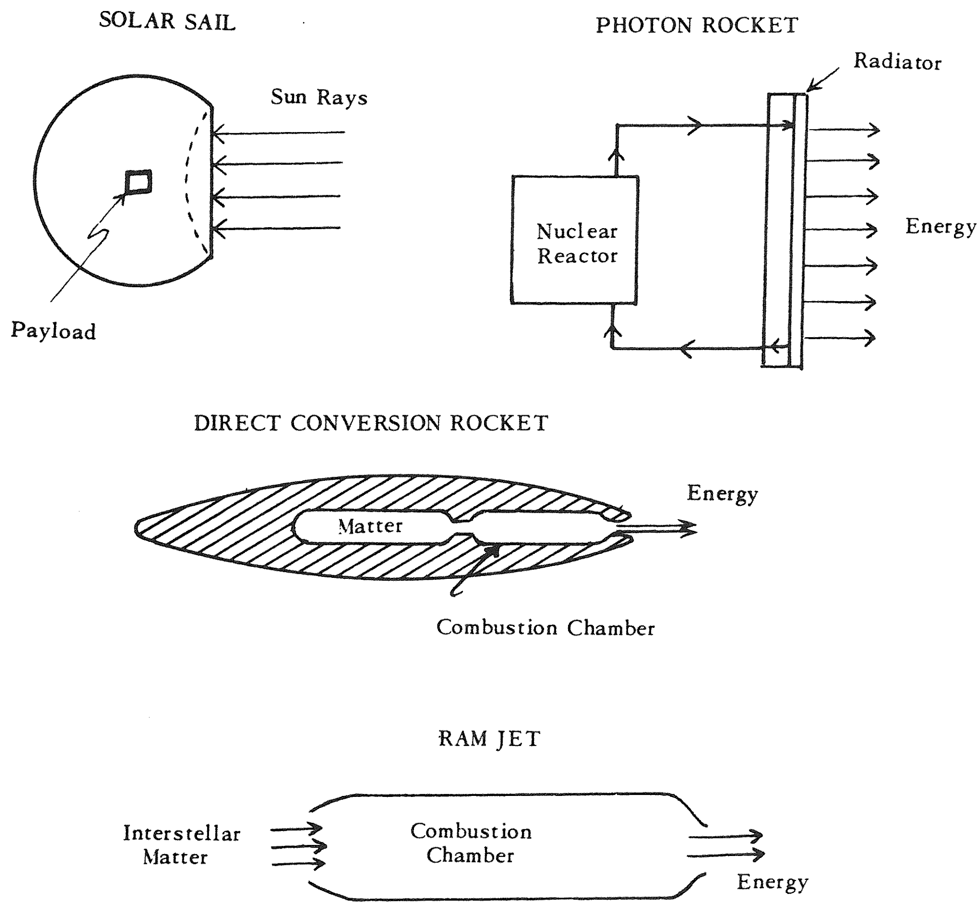


Figure 28. Photon propulsion concepts.

PHOTON PROPULSION—Another idea that may prove feasible in the future is the photon concept. Four approaches for using photon propulsion are shown in Figure 28.

SUMMARY

This chapter presented the basic laws of rocket operation, and some of the terminology and definitions pertaining to rocket engine performance. It also discussed the interaction of design and sizing parameters of rocket vehicles, and methods of increasing rocket vehicle performance. Some of the advanced pro-

TABLE 5

Propulsion Systems

Type	*Specific Impulse, sec.*
High Thrust — To — Weight Ratio	
Chemical	To 600
Nuclear	800–2,000
Low Thrust — To — Weight Ratio	
Electric	2,000–30,000
Fusion	1,000,000
Photon	30,000,000

pulsion concepts were also reviewed. A comparison of the relative specific impulse values for all of these is shown in Table 5.

Although the various types in Table 5 are compared on the basis of specific impulse, it alone does not present the complete picture. Only chemical propellants and high thrust nuclear rockets have thrust-to-weight ratios that are sufficient to permit launches from the earth. Of course, this also means a launch potential from other bodies, such as the moon, Mars, and Venus which have gravitational fields. Others (electric, and photon) are applicable to missions in low gravitational fields. These fields could be in deep space or between one planet's orbit and another planet's orbit. Again, a vehicle with chemical or nuclear reactor engines would be needed to travel from the orbital vehicle to a planet or moon surface, and to return to the orbital vehicle.

Various applications of engines for the immediate future are summarized in Figure 29.

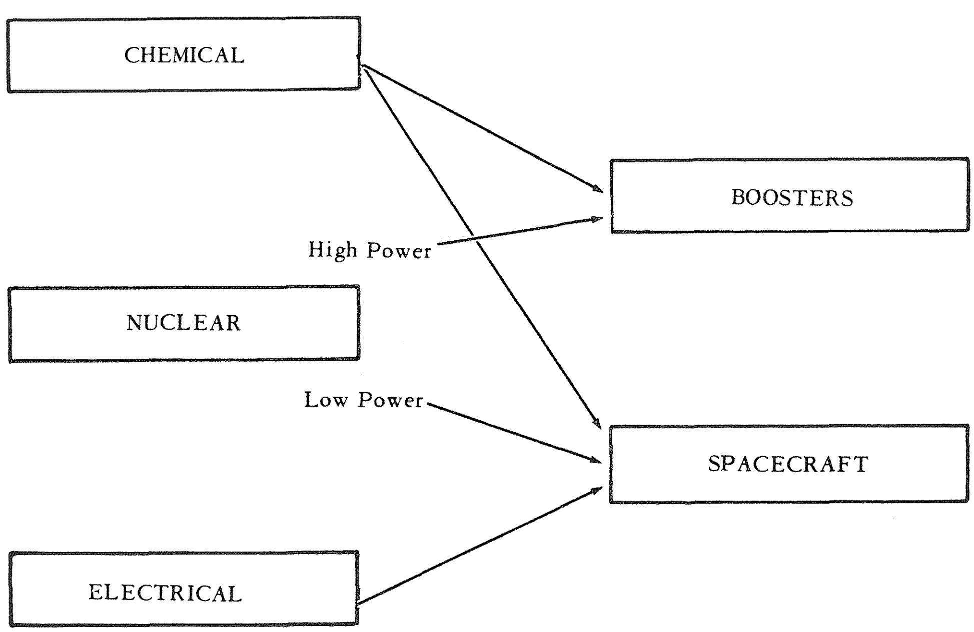


Figure 29. Types of propulsion and their applications.

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PROPULSION SYMBOLS

a—acceleration
A_e—nozzle exit area
A_t—nozzle throat area
C_p—constant pressure specific heat
C_v—constant volume specific heat
F—thrust
g—acceleration of gravity
I_d—density impulse
I_{sp}—specific impulse
I_t—total impulse
k—ratio of specific heats
$1n$—natural logarithm
m—molecular weight
M—mass
$\dot{M}$—mass rate of flow
MR—mass ratio
p—electric power
P_c—combustion chamber pressure
P_e—exhaust pressure
P_o—ambient pressure
Q—reactor thermal power
r—mixture ratio
t_b—burning time
T_c—combustion chamber temperature
T_e—exit temperature
T_i—inlet temperature
V—volume
Δv_a—actual velocity change
Δv_i—ideal velocity change
Δv_l—velocity change losses
v_e—nozzle exit velocity
v_r—earth surface velocity
W—weight
$\dot{W}$—weight rate of flow
$\dot{W}_f$—weight rate of fuel flow
$\dot{W}_o$—weight rate of oxidizer flow
W_p—propellant weight
W_1—vehicle weight at engine start
W_2—vehicle weight at engine shutdown
ϵ—nozzle expansion ratio
Ψ—thrust-to-weight ratio
η—electrical efficiency
ν—heat transfer efficiency

SOME USEFUL PROPULSION EQUATIONS

1. Thrust of rocket: $F = \frac{\dot{W}}{g} v_e + A_e (P_e - P_o)$
2. Expansion ratio: $\epsilon = \frac{A_e}{A_t}$
3. Measured specific impulse: $I_{sp} = \frac{F}{\dot{W}_p}$
4. Mass ratio: $MR = \frac{W_1}{W_2} = \frac{\text{Engine start weight}}{\text{Engine stop weight}}$
5. Overall mass ratio: $MR = (MR_1) \times (MR_2) \times MR_3) \times \ldots$
6. Thrust-to-weight ratio: $\Psi = \frac{F}{W}$
7. Lift-off acceleration: $a = (\Psi - 1)$ g's
8. Ideal velocity change: $\Delta v_i = I_{sp}\, g\, 1nMR$
9. $\Delta v_a = \Delta v_i - \Delta_l + v_r$
10. Theoretical specific impulse:

$$I_{sp} = 9.797\sqrt{\left(\frac{k}{k-1}\right)\left(\frac{T_c}{m}\right)\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]}$$

11. Mixture ratio: $r = \frac{\dot{W}_o}{\dot{W}_f}$
12. Density impulse: $I_d = (I_{sp}) \times (SG)$

13. Total impulse: $I_t = (F) \times (t) = (I_{sp})\ (W_p)$
14. Total Δv (multistage vehicle): $\Delta v_t = \Delta v_1 + \Delta v_2 + \Delta v_3 + \ldots$
15. Thrust of hydrogen-fueled nuclear rocket:

$$F = 6.94 \sqrt{\dot{W}\,[947\, \nu\, Q - 3.76\, \dot{W}\, (T_e - T_i)]}$$ (See Appendix E)

16. Thrust of electric rocket: $F = 38.4 \sqrt{\eta p\, \dot{M}}$ (See Appendix E)

Appendix B

DETERMINATION OF THE ANGLE BETWEEN TWO ORBITAL PLANES

FOR SPACE rendezvous to occur both vehicles must simultaneously be in the same orbital plane and be at the same location in identical orbits.[1] For this discussion the orbit requirement will be assumed to have been satisfied independently of the orbital plane requirement. The maneuvering operations necessary to satisfy both requirements may occur in any sequence; however, only the orbital plane requirement will be discussed here.

In space rendezvous operations one encounters the problem of reaching a specified orbital plane from either another orbital plane or a specific launch site. Determination of the plane change angle (α) required to change from one orbital plane to another, at the intersection of the two planes, is complicated only by the requirement to use spherical trigonometry instead of the plane trigonometry to which we are more accustomed.

To simplify the discussion of the solution, the orbital plane of the vehicle is defined as the initial plane, and the desired new orbital plane is defined as the final plane. Since any launch must initially enter an orbital plane which is dependent on the launch site latitude (L), launch azimuth (β), and time and date of launch, the above definitions will suffice.

For the case of an initial plane and a final plane, the solution is dependent on knowing or estimating the right ascension (Ω) and inclination (i) angles of the two planes. For the case of a final plane and a launch site, the solution is dependent on: the Ω and i angles for the plane; time between launch and passage of the orbital plane over the launch site; and the launch latitude and azimuth.

Two examples below illustrate the mathematics involved and the approximation techniques available for obtaining solutions. In the first example it will be shown that only the difference in right ascension angles ($\Delta\Omega$) is required; the specific values of Ω need not be known.

Case I

A vehicle in orbit 1 is to be maneuvered into the plane of orbit 2. To accomplish this maneuver the value of the plane change angle is required.

Orbital Parameters

	Orbit 1	Orbit 2
Ω	30°	45°
i	28°	30°

[1] Identical orbits occur when the magnitude and orientation of the major axes (2a) are the same and the eccentricities (ϵ) are equal.

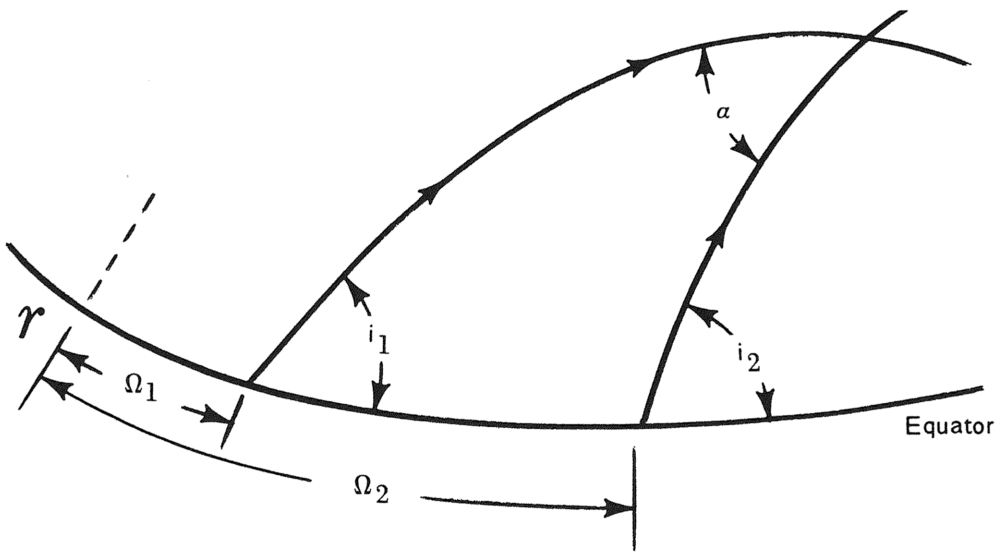
From spherical trigonometry the plane change angle can be determined from:

$$\cos\left(\frac{\alpha}{2}\right) = \frac{\cos\frac{i_2 - i_1}{2}}{\cos(x)} \cos\frac{\Omega_2 - \Omega_1}{2}$$

Where the angle x, a dummy angle used in the calculation, is determined from:

$$\tan(x) = \frac{\cos\frac{i_2 + i_1}{2}}{\cos\frac{i_2 - i_1}{2}} \tan\frac{\Omega_2 - \Omega_1}{2}$$

The geometry of the problem is as shown below.



From the equations and the geometry it is apparent that: only the difference ($\Delta\Omega$) in Ω is required; it will always be positive; and it can be reduced to an angle between 0 and 180 degrees. Therefore, for this problem $\Delta\Omega$ is 15°.

Substituting to determine the angle x:

$$\tan x = \frac{\cos\frac{30° + 28°}{2}}{\cos\frac{30 - 28}{2}} \tan\frac{15°}{2} = \frac{\cos(29°)}{\cos(1°)} \tan(7.5°)$$

$$= \frac{0.8746197}{0.9998477} \times (0.131652) = 0.1151629$$

Therefore, $x = 6° \, 34'$ and $\cos x = 0.9934727$

Substituting to determine the plane change angle α:

$$\cos\left(\frac{\alpha}{2}\right) = \frac{\cos\dfrac{30^\circ - 28^\circ}{2}}{0.9934727}\cos(7.5^\circ)$$

$$= \frac{0.9998477}{0.9934727} \times (0.9914449) = 0.997807$$

Therefore, $\frac{\alpha}{2} = 3^\circ\ 48'$ or $\alpha = 7^\circ\ 36'$.

The required plane change angle is $7^\circ\ 36'$.

Case II

A spacecraft on a launch vehicle at Cape Kennedy ($L = 28^\circ$) must attain orbit in a plane which has a southwest-northeast orientation and an inclination angle of 30°. Ideally, the vehicle would be launched and injected along the required azimuth angle, when the desired orbital plane was directly over the launch site. This is, however, a very difficult timing problem. The spacecraft/launch vehicle combination has limited velocity change capabilities which narrows the "launch window." A launch occurring within this window of time will permit injection into an initial parking orbit and a subsequent plane change maneuver to the desired orbital plane.

Because of launch delays the launch site passed through the desired final orbital plane 8 minutes prior to lift-off. Since the launch vehicle guidance was programmed prior to launch, the spacecraft will be injected into an initial parking orbit, and then it must make a plane change to the desired final orbital plane. The programmed launch azimuth angle is 79°. What is the value of the plane change angle required to complete the mission?

The apparent unknowns are the initial orbit inclination angle, and the $\Delta\Omega$.

To find the inclination angle of this parking orbit, use the equation for inclination angle of the launch:

$$\cos i = \cos(L)\sin(\beta) = \cos(28^\circ)\sin(79^\circ) = 0.8660$$

Therefore, $i = 30^\circ$ for the parking orbit.

It is possible to approximate $\Delta\Omega$ using the earth's rate of rotation.

$$\text{Rotation Rate} = \frac{360^\circ}{24\text{ hrs}} = \frac{360^\circ}{(24 \times 60)\text{ min}} = 0.25^\circ/\text{min}$$

For the situation of equal inclination angles, the $\Delta\Omega$ is equal to the time between launch and passage of the final orbital plane overhead, multiplied by the earth's rotation rate:[2]

$$\Delta\Omega = 0.25^\circ/\text{min} \times 8\text{ min} = 2^\circ$$

By making use of the same equations used for Case I, the plane change angle α will be approximately 1°.

[2] For the case of non equal inclination angles it is necessary to add a term $\Delta\Omega^*$.

$$\Delta\Omega^* = \Omega^*_1 \pm \Omega^*_2 \quad \begin{cases} - & \text{Both inclination angles in the same quadrant} \\ + & \text{Inclination angles in different quadrants} \end{cases}$$

$\Delta\Omega^*$ must be positive
To find the Ω^*'s use:

$\sin\Omega^*_1 = \tan(L)\cot(i_1)$
$\sin\Omega^*_2 = \tan(L)\cot(i_2)$

Then the total $\Delta\Omega$ equals the $\Delta\Omega$ due to the earth's rotation plus the $\Delta\Omega^*$ due to the difference in inclination angles.

List of Symbols

α—Plane Change Angle
β—Azimuth Angle
$\Delta\Omega$—Difference in Right Ascension Angles
Ω—Right Ascension Angle
i—Inclination Angle
L—Angle of Latitude
x—Dummy Angle

Subscripts

1. Left hand orbital plane
2. Right hand orbital plane

Superscripts

* Dummy Variable

Appendix C

BALLISTIC MISSILE TRAJECTORIES

THE PURPOSE of this section is to present the quantitative analysis and geometry of ballistic missile trajectories. The analysis is based upon the two-body problem with the earth as the large, central body. Specific problems concern the range of a ballistic missile for specified burnout conditions and the effect of errors in burnout conditions on the desired impact point.

In order to simplify the problem so that the fundamentals are highlighted, a series of practical assumptions are made. The earth is assumed to be spherical and non-rotating. The effect of the earth's oblateness is very complicated; an extensive study is beyond the scope of this text. There is an oblateness effect upon the trajectory itself; the target, therefore, is not in the same position that it would be on a spherical earth. The effect of the earth's oblateness is more pronounced upon satellite trajectories than upon ballistic missile trajectories because of the longer time of flight. The powered and reentry portions of an actual trajectory are not considered here; the two-body problem cannot be applied to these parts of the trajectory, as energy is dissipated in air friction in both parts, and propulsive energy is being added to the system during the powered trajectory; thus, the conditions of constant mechanical energy and constant angular momentum do not apply during these portions of the trajectory.

Geometry and Standard Values of Earth Parameters

Before proceeding to computational analysis, it is essential that the geometry of the ballistic missile problem be understood. The geometry is presented in Fig. 1. The symbols are defined below. Numerical calculations require the use of earth parameters, and standard values of these are also presented below.

r_{bo}	radius to the burnout point
v_{bo}	magnitude of the velocity (speed) at the burnout point
ϕ_{bo}	angle between the velocity vector (tangent to the trajectory) and the local horizontal at the burnout point
θ	angle from the point of apogee, measured in the direction of motion
Λ	total range angle
Γ	powered trajectory range angle
Ω	reentry range angle
Ψ	free-flight range angle
r_e	radius of the earth $= 3440$ NM $= 20.9 \times 10^6$ feet
g_o	acceleration of gravity at sea level $= 32.2 \ \frac{\text{ft}}{\text{sec}^2}$
μ	defined as Gm_1, where G is the Universal Gravitational Constant and m_1 is the mass of the earth, $\mu = 14.08 \times 10^{15} \ \frac{\text{ft}^3}{\text{sec}^2}$

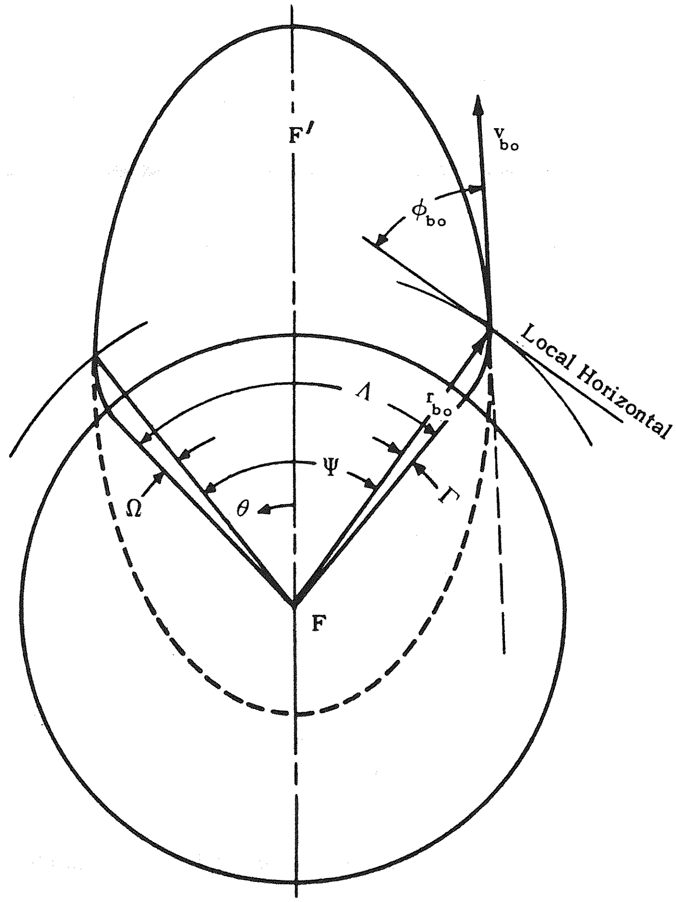


Figure 1. The ballistic missile trajectory.

General Ballistic Missile Problem

If a launch point and target for a ballistic missile are selected, then only the range is known. If the burnout conditions of v_{bo}, r_{bo} and ϕ_{bo} are given, the trajectory equation is uniquely determined, and a definite range will be obtained. Actually, there are many possible combinations of these four parameters, and no good basis for a choice exists until some other condition is imposed. One such condition might be to specify that v_{bo} be the smallest that will get the payload to the target. The converse of this statement is to require maximum range for a given v_{bo}. Clearly, a relationship between range and the trajectory parameters is needed and must be found.

The Range Equation

Symmetrical trajectories ($h_{bo} = h_{re}$) involve simpler mathematics than do unsymmetrical trajectories ($h_{bo} \neq h_{re}$). Because the symmetrical trajectories adequately demonstrate the principles involved, this treatment of ballistic missiles will be limited to symmetrical trajectories. By using the polar equation for the two-body trajectory and suitable geometric substitutions, the ballistic missile range equation can be developed in terms of Ψ (the free flight range angle) and the burnout conditions (v_{bo}, r_{bo}, and ϕ_{bo}). It is given here as:

$$\cot \frac{\Psi}{2} = \frac{2\mu}{v^2_{bo}\, r_{bo}} \csc 2\phi_{bo} - \cot \phi_{bo} \qquad (1)$$

Equation (1) can be simplified by substitution of the following convenient definition:

$$\text{let } Q = \frac{v^2 r}{\mu} \qquad (2)$$

$$\text{then } Q_{bo} = \frac{v^2_{bo}\, r_{bo}}{\mu}$$

The parameter Q has interesting characteristics. It is equal to 1 for circular orbital conditions, and it is equal to 2 for minimum escape conditions. With this substitution, equation (1) becomes

$$\cot \frac{\Psi}{2} = \frac{2}{Q_{bo}} \csc 2\phi_{bo} - \cot \phi_{bo} \qquad (3)$$

The following example problem will illustrate how (1) may be used.

Example Problem

The following information is given about a ballistic missile:

$$v_{bo} = 16{,}000 \text{ ft/sec}$$
$$\phi_{bo} = 21^\circ$$
$$h_{bo} = 164.5 \text{ NM}$$

What is the free-flight ground range of this missile on a spherical, non-rotating earth?

Problem Solution

Seven graphs, Figs 4-10, are included at the end of this section for use in graphical solutions.

Graphical: Graph Fig. 4, $Q_{bo} = .4$

Graph Fig. 7, $R_{ff} = 1445$ NM

Calculated:

$$Q_{bo} = \frac{v_{bo}^2\, r_{bo}}{\mu} = \frac{(1.6 \times 10^4 \text{ ft/sec})^2 \times 21.9 \times 10^6 \text{ ft}}{14.08 \times 10^{15} \text{ ft}^3/\text{sec}^2}$$

$$Q_{bo} = .398 \approx .4$$

$$\cot \frac{\Psi}{2} = \frac{2}{Q_{bo}} \csc 2\phi_{bo} - \cot \phi_{bo} = \frac{2}{.4} \csc 42^\circ - \cot 21^\circ$$

$$\cot \frac{\Psi}{2} = \frac{2 \times 1.49}{.4} - 2.6 = 4.84$$

$$\frac{\Psi}{2} = 11.7^\circ$$

$$\Psi = 23.4^\circ$$

$$R_{ff} = 23.4^\circ \times 60 \text{ NM}/^\circ = 1404 \text{ NM}$$

The Equation for Flight Path Angle

A more practical ballistic missile problem results from defining a launch point, target, and missile. The launch point and target determine the range (Ψ) while the missile is capable of attaining a certain burnout velocity and height. In this problem, then, Ψ, v_{bo}, and r_{bo} are given and it is desired to find ϕ_{bo}. Equation (1) may be solved for ϕ and be refined to the following form:

$$\cot \phi_{bo} = \frac{-\cot \frac{\Psi}{2} \pm \sqrt{\cot^2 \frac{\Psi}{2} + \frac{4}{Q_{bo}}\left(1 - \frac{1}{Q_{bo}}\right)}}{2\left(1 - \frac{1}{Q_{bo}}\right)} \tag{4}$$

Equation (4) points out some very important facts. When $Q_{bo} < 1$, Ψ must be less than 180°, and there are two values of ϕ_{bo} for a given range, v_{bo}, and r_{bo}. There are two trajectories, then, to the target. The trajectory corresponding to the larger value of flight path angle is called the "high" trajectory; the trajectory associated with the smaller flight path angle is the "low" trajectory.

A simple illustration of the principle involved is the behavior of water discharged from a garden hose. With a constant water pressure and nozzle setting, the velocity of the water leaving the nozzle is fixed. If a target well within the reach of the water is selected, the target can be hit by both a high and a low trajectory.

The high and low trajectories are sometimes referred to as "lofted" and "flat" trajectories, respectively.

When $1 \leq Q_{bo} < 2$, and $\Psi < 180°$, there is still a high trajectory, but the low trajectory does not exist because it would penetrate the earth. Interestingly enough, however, when $1 < Q_{bo} < 2$, Ψ can be greater than 180°. An illustration of such a trajectory would be a missile directed at the Asian continent from North America via the south polar region. Although such a trajectory would be capable of avoiding a northern radar "fence," it would be costly in terms of payload delivered.

It is interesting to compare parameters of the high and low trajectories. As v_{bo} and r_{bo} are the same for the two trajectories, E is also the same. But $a = -\mu/2E$, so the major axes of the trajectories are the same. The angular momentum is smaller for the high trajectory; c and ϵ are larger. It is also evident that the time-of-flight on the high trajectory will be longer.

For the constraint of fixed energy, it is reasonable to anticipate that there are certain ranges which are attainable and certain ranges which are not attainable; between these two groups there is the limiting case in which the range is just barely attainable.

The usual way to find such a maximum is to solve for the dependent variable (in this case the range, Ψ) in terms of the independent variable (in this case, ϕ_{bo}). The partial derivative of the dependent variable with respect to the independent variable is then formed and set equal to zero to determine the conditions for a maximum (or a minimum). In our case the range equation expresses Ψ in terms of ϕ_{bo} and the partial of Ψ with respect to ϕ_{bo} when set equal to zero is:

$$\frac{\partial \Psi}{\partial \phi_{bo}} = \frac{2 \sin(\Psi + 2\phi_{bo})}{\sin 2\phi_{bo}} - 2 = 0$$

Hence

$$\sin(\Psi + 2\phi_{bo}) - \sin 2\phi_{bo} = 0 \tag{5}$$

Using the trigonometric identity

$$\sin x - \sin y = 2 \sin \tfrac{1}{2}(x - y) \cos \tfrac{1}{2}(x + y)$$

equation (5) becomes

$$2 \sin \Psi/2 \cos \tfrac{1}{2}(\Psi + 4\phi_{bo}) = 0 \tag{6}$$

Equation (6) expresses the conditions under which a maximum or a minimum will occur. One condition occurs when $\sin \Psi/2 = 0$. This implies that Ψ is either 0 or 2π, neither of which is of interest for a ballistic missile. The case of interest occurs when

$$\cos \tfrac{1}{2}(\Psi + 4\phi_{bo}) = 0$$

or

$$\tfrac{1}{2}(\Psi + 4\phi_{bo}) = \pi/2$$

from which

$$\phi_{bo} = \tfrac{1}{4}(\pi - \Psi) \tag{7}$$

Equation (7) states the relationship which must exist between Ψ and ϕ_{bo} on a maximum range trajectory. It is important to observe that there is one and only one value of ϕ_{bo} for a maximum range trajectory; the maximum range trajectory is unique.

ECCENTRICITY EQUATION.—It is inconvenient to calculate ϵ from the equation, $\epsilon = \sqrt{1 + \frac{2EH^2}{\mu^2}}$, when given the range and burnout conditions of a ballistic missile. Eccentricity, as a function of range and burnout conditions, would enable more rapid calculations. Two forms of the eccentricity equation are given here as a convenience:

$$\epsilon = \frac{\sin \phi_{bo}}{\sin\left(\phi_{bo} + \frac{\Psi}{2}\right)} \tag{8}$$

$$\epsilon = \sqrt{Q^2_{bo} \cos^2 \phi_{bo} - 2Q_{bo} \cos^2 \phi_{bo} + 1} \tag{9}$$

Range and Azimuth Errors

Still considering the earth to be non-rotating, the target, launch (or burnout) point, and center of the earth define a plane in which the trajectory, and thus the velocity vector, must lie. The burnout velocity vector can have errors in the intended plane as follows: position, magnitude, and flight path angle. It is also possible for the burnout velocity vector to have errors out of the intended plane in which case a new plane is defined. The most important error out of the intended plane is azimuth error. It will be considered before the errors in the intended plane, as the treatment can be brief.

AZIMUTH ERROR.—Figure 2 illustrates the geometry of azimuth error. The arc length, a, represents the cross-range error. The arc lengths, b and c, are equal to each other and are the range. It is usual in spherical trigonometry to measure arc lengths in terms of the angle subtended at the center of the sphere. Using this convention, $b = c = \Psi$, the range angle defined earlier. The law of cosines in spherical trigonometry is:

$$\cos a = \cos b \cos c + \sin b \sin c \cos A$$

which becomes:

$$\cos a = \cos^2 \Psi + \sin^2 \Psi \cos A$$

As A is a small angle, cos A can be approximated by $1 - \frac{A^2}{2}$; then,

$$\cos a = \cos^2 \Psi + \sin^2 \Psi - \frac{A^2 \sin^2 \Psi}{2} = 1 - \frac{A^2 \sin^2 \Psi}{2}$$

But a is also a small angle so that cos a can be approximated by $1 - \frac{a^2}{2}$; substituting this,

$$1 - \frac{a^2}{2} = 1 - \frac{A^2 \sin^2 \Psi}{2}$$

And:

$$a = A \sin \Psi \qquad (10)$$

Equation (10) is in terms of angular measure; in order to have the cross-range error in linear dimensions, the radius of the earth must be multiplied by the central angle, a, if it is in radians; if it is in degrees, the relationship that 60NM on the surface are equivalent to a 1 degree central angle at the center of the earth may be used.

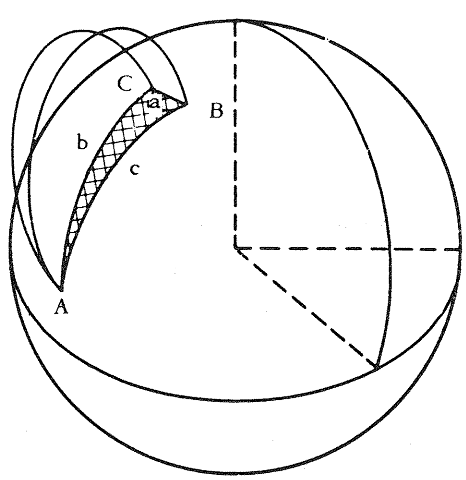


Figure 2. Geometry of azimuth error.

quantity $\frac{\partial\Psi_{bo}}{\partial\phi_{bo}}$. Thus, the effects of each of the possible contributors to free flight range error are influenced by the factors $\frac{\partial\Psi}{\partial v_{bo}}$, $\frac{\partial\Psi}{\partial r_{bo}}$, and $\frac{\partial\Psi}{\partial\phi_{bo}}$. They are referred to as "influence coefficients," since these factors exert an influence on the size of the range error resulting from burnout errors.

There are three types of problems that can be defined in terms of the influence coefficients and equation (12). (1) Given the guidance system capabilities [Δv_{bo}, Δr_{bo} and $\Delta\phi_{bo}$] and the influence coefficients, find the range error. (2) Given the allowable range error [$\Delta\Psi$] and the influence coefficients, find the allowable burnout errors [Δv_{bo}, Δr_{bo}, $\Delta\phi_{bo}$]. (3) Given the allowable range error [$\Delta\Psi$] and the guidance system capabilities [Δv_{bo}, Δr_{bo}, $\Delta\phi_{bo}$], find a trajectory which yields an appropriate set of influence coefficients. Type 3 is by far the most difficult problem since the first two types imply that the trajectory is known. In any case, inseparably related to the study of range errors is the evaluation of the influence coefficients. All of the influence coefficients are obtained by differentiating the range equation:

$$\cot\frac{\Psi}{2} = \frac{2\mu}{v^2_{bo}\, r_{bo}}\csc 2\phi_{bo} - \cot\phi_{bo}$$

In particular the influence coefficient for burnout flight path angle is obtained by taking the partial derivative with respect to ϕ_{bo}.

$$\frac{\partial\Psi}{\partial\phi_{bo}} = \frac{2\sin(\Psi + 2\phi_{bo})}{\sin 2\phi_{bo}} - 2 \tag{13}$$

Thus, range and flight path angle determine the influence coefficient for errors in flight path angle. Figure 8 is a plot of this influence coefficient for specific values of Q at various free flight ranges. For convenience in computation the values of the influence coefficient are plotted in *NM* range error per degree error in the flight path angle alignment.

The burnout speed error influence coefficient is obtained by differentiating the range equation with respect to v_{bo}:

$$\frac{\partial\Psi}{\partial v_{bo}} = \frac{8\mu}{v^3_{bo}\, r_{bo}}\;\frac{\sin^2\frac{\Psi}{2}}{\sin 2\phi_{bo}} \tag{14}$$

For any range, and ϕ_{bo} less than 90°, $\frac{\partial\Psi}{\partial v_{bo}}$ will be positive. Hence, a positive speed error will increase the range. If the actual burnout speed is greater than the intended value, the missile will overshoot the target. Conversely, if the actual burnout speed is less than the intended value, the missile will always fall short of the target.

The influence coefficient is determined by the trajectory requirements of range, v_{bo}, r_{bo}, and ϕ_{bo}. Δv is the factor controlled by the guidance system. Figure 9 is a plot of this influence coefficient for specific values of Q at various free flight ranges. For convenience in computation the values of the influence coefficient are plotted in *NM* range error *per ft/sec* error in the velocity magnitude at burnout.

The influence coefficient is smaller for the high trajectory resulting in less error at the target for a given speed error. Some missiles have guidance systems which are accurate enough to permit use of the low trajectory at ICBM range.

From equation (10), it can be seen that, when the range angle, Ψ, is 90 degrees, maximum error occurs. Also, when Ψ is 180 degrees, the azimuth error returns to zero.

RANGE ERROR AS A FUNCTION OF ERRORS IN THE TRAJECTORY PLANE.—In this section it is assumed that the burnout velocity vector lies in the intended trajectory plane (hence the cross-range error is zero), but the possibility exists that the velocity vector may have errors in position, magnitude and flight path angle. To trace the possible sources of range errors, it is helpful to write down the equation for the total range angle (Λ) in terms of the powered range angle, (Γ), the free flight range angle (Ψ), and the reentry range angle (Ω). This equation is simply:

$$\Lambda = \Gamma + \Psi + \Omega$$

Since Λ is the sum of its three parts, any change in the total range angle ($\Delta\Lambda$) can be traced to three possible sources. That is:

$$\Delta\Lambda = \Delta\Gamma + \Delta\Psi + \Delta\Omega \tag{11}$$

Since the changes ($\Delta\Lambda$, $\Delta\Gamma$, $\Delta\Psi$, $\Delta\Omega$) in the last equation represent departures from the desired trajectory, they may properly be termed range errors. An error committed during any portion of the trajectory contributes directly to the total range error $\Delta\Lambda$. The three possible range errors are not independent, but each contributes to the total range error, and none can be ignored.

Concentrate attention upon the free flight range error $\Delta\Psi$ and trace the possible source of $\Delta\Psi$. The range equation, equation (1), expresses the free flight range as a function of the burnout conditions, v_{bo}, r_{bo}, and ϕ_{bo}. Following the rules for the formation of the total differential of an explicit function:

$$d\Psi = \frac{\partial\Psi}{\partial v_{bo}} dv_{bo} + \frac{\partial\Psi}{\partial r_{bo}} dr_{bo} + \frac{\partial\Psi}{\partial\phi_{bo}} \phi_{bo}$$

This is an equation for the total differential, $d\Psi$. The equation for the change in free flight range, $\Delta\Psi$, reads:

$$\Delta\Psi \cong \frac{\partial\Psi}{\partial v_{bo}} \Delta v_{bo} + \frac{\partial\Psi}{\partial r_{bo}} \Delta r_{bo} + \frac{\partial\Psi}{\partial\phi_{bo}} \Delta\phi_{bo} \tag{12}$$

$\Delta\Psi$ is the free flight range error. The quantities, Δ_{bo}, Δr_{bo}, $\Delta\phi_{bo}$ represent departure from the desired burnout conditions and so represent the burnout speed error, the burnout altitude error, and the burnout flight path angle error, respectively.

Next, consider the three quantities $\frac{\partial\Psi}{\partial v_{bo}}, \frac{\partial\Psi}{\partial r_{bo}}$, and $\frac{\partial\Psi}{\partial\phi_{bo}}$. In equation (11) each of the possible errors enters the summation in a one-to-one manner while in equation (12) the possible errors are multiplied by the factors $\frac{\partial\Psi}{\partial v_{bo}}, \frac{\partial\Psi}{\partial r_{bo}}$, and $\frac{\partial\Psi}{\partial\phi_{bo}}$. This fundamental difference means, for example, that a burnout flight path angle error of one degree will not necessarily produce a range error of one degree but will produce an error which depends upon the size of the

The last influence coefficient of interest related errors in altitude (radius) to the resulting range errors. Here the missile is considered to have attained the proper burnout velocity on the intended radial, but at an incorrect radius.

$$\frac{\partial \Psi}{\partial r_{bo}} = \frac{4\mu}{v^2_{bo}\, r^2_{bo}} \frac{\sin^2 \dfrac{\Psi}{2}}{\sin 2\phi_{bo}} \tag{15}$$

Figure 10 is a plot of this influence coefficient for specific values of Q at various flight ranges. For convenience in computation the values of the influence coefficient are plotted in *NM* range error *per* 10^3 *ft* error in radius (or altitude) at burnout.

Again, the error is not as great for the high trajectory. The range error resulting from an error in burnout height should not be confused with the change in range which would be obtained in launching a missile from a high elevation. If an Atlas, say, were launched from the top of Pikes Peak, it would have a range increase due to $\frac{\partial \Psi}{\partial r_{bo}}$, but it would have another, probably greater, range increase resulting from its powered trajectory occurring in less dense atmosphere. The same missile launched at sea level would have to expend considerable energy overcoming the large drag forces up to 14,000 ft. The effect of a launch altitude then is primarily related to the powered trajectory and is a separate phenomenon from the influence coefficient effect.

Example Problem

Telementry from Cape Kennedy indicated that an ICBM achieved burnout conditions of h_{bo} = 164.5 NM, v_{bo} = 22,700 ft/sec, and $\phi_{bo} = 30°$.

(a) The particular guidance system used had an accuracy of $\Delta\, h_{bo} = \pm$ 1000 ft, $\Delta\, v_{bo} = \pm$ 2 ft/sec, and $\Delta\, \phi_{bo} = \pm .2°$. ($3.49 \times 10^{-3}$ radians). What is the total free-flight range error? *Ans.* Q = .8, R_{ff} = 4900 NM; $\Delta\Psi = -3.99$ NM error.

(b) An improved guidance system which is able to control $\Delta\phi = \pm .05°$ (8.72×10^{-4} radians) is installed. What is the total free-flight error now? *Ans.* $\Delta\Psi$ = .51 NM error.

(c) Assuming $\Delta\, h_{bo}$ and $\Delta\, \phi_{bo}$ are both positive, what correction in v_{bo} ($\Delta\, v_{bo}$) would be necessary to zero the free-flight range error found in (b) above? *Ans.* $\Delta\, v_{bo}$ = 1.4 ft/sec.

Problem Solution

a. *Graphical:* Graph Fig. 5, Q = .8 (by interpolation) *Ans.*

Graph Fig. 7, R_{ff} = 4900 NM (High trajectory) *Ans.*

Graph Fig. 8, $\frac{\partial \Psi}{\partial \phi_{bo}} = -30$ NM/deg

Graph Fig. 9, $\frac{\partial \Psi}{\partial v_{bo}} = 0.8$ NM/ft/sec

Graph Fig. 10, $\frac{\partial \Psi}{\partial r_{bo}} = .41$ NM/10^3 ft

$$\Delta\Psi = \left(\frac{\partial \Psi}{\partial \phi_{bo}} \Delta\phi_{bo}\right) + \left(\frac{\partial \Psi}{\partial v_{bo}} \Delta v_{bo}\right) + \left(\frac{\partial \Psi}{\partial r_{bo}} \Delta r_{bo}\right)$$

Arbitrarily assuming all injection errors are positive:

$$\Delta\Psi = \left(-30 \frac{\text{NM}}{\text{deg}} \times 0.2 \text{ deg}\right) + \left(0.8 \frac{\text{NM}}{\text{ft/sec}} \times 2 \text{ ft/sec}\right) + \left(.41 \frac{\text{NM}}{10^3 \text{ ft}} \times 10^3 \text{ ft}\right)$$

$$\Delta\Psi = (-6.0 \text{ NM} + 1.6 \text{ NM} + .41 \text{ NM}) = -4.0 \text{ NM} \qquad \textit{Ans.}$$

a. *Calculated:*

$$Q = \frac{v_{bo}^2 r_{bo}}{\mu} = \frac{(2.27 \times 10^4 \text{ ft/sec})^2 (20.9 \times 10^6 + 164.5 \times 6080) \text{ ft}}{14.08 \times 10^{15} \text{ ft}^3/\text{sec}^2}$$

$$Q = \frac{(5.14 \times 10^8)(21.9 \times 10^6)}{14.08 \times 10^{15}} = .8 \qquad \textit{Ans.}$$

$$\cot \frac{\Psi}{2} = \frac{2}{Q_{bo}} \csc 2\phi_{bo} - \cot \phi_{bo} = \frac{2}{.8} \csc 60° - \cot 30°$$

$$\cot \frac{\Psi}{2} = 2.5\,(1.155) - 1.732 = 2.888 - 1.732 = 1.156$$

$$\frac{\Psi}{2} = 40.8° \qquad \Psi = 81.6°$$

$$R_{ff} = 81.6° \times 60 \text{ NM}/° = 4896 \text{ NM} \qquad \textit{Ans.}$$

$$\frac{\partial\Psi}{\partial\phi_{bo}} = \frac{2 \sin(\Psi + 2\phi_{bo})}{\sin 2\phi_{bo}} - 2 = \frac{2 \sin(81.6° + 60°)}{\sin 60°} - 2$$

$$\frac{\partial\Psi}{\partial\phi_{bo}} = \frac{2 \sin 141.6°}{\sin 60°} - 2 = \frac{2 \sin 38.4°}{\sin 60°} - 2 = \frac{2\,(.621)}{.866} - 2$$

$$\frac{\partial\Psi}{\partial\phi_{bo}} = \frac{1.242}{.866} - 2 = 1.434 - 2 = -.566$$

$$\frac{\partial\Psi}{\partial v_{bo}} = \frac{8\mu}{v_{bo}^3 r_{bo}} \frac{\sin^2 \frac{\Psi}{2}}{\sin 2\phi_{bo}} \qquad \frac{8 \times 14.08 \times 10^{15}}{11.70 \times 10^{12} \times 21.9 \times 10^6} \times \frac{.653^2}{.866}$$

$$\frac{\partial\Psi}{\partial v_{bo}} = \left(\frac{112.6 \times 10^{-3}}{256.2}\right)(.492) = (.44 \times 10^{-3})(.492) = 2.16 \times 10^{-4} \frac{\text{rad}}{\text{ft/sec}}$$

$$\frac{\partial\Psi}{\partial r_{bo}} = \frac{4\mu}{v_{bo}^2 r_{bo}^2} \frac{\sin^2 \frac{\Psi}{2}}{\sin 2\phi_{bo}} = \frac{4 \times 14.08 \times 10^{15}}{5.15 \times 10^8 \times 4.80 \times 10^{14}} \times \frac{.653^2}{.866}$$

$$\frac{\partial\Psi}{\partial r_{bo}} = \frac{56.3 \times 10^{-7}}{24.72}(.492) = (2.28 \times 10^{-7})(.492) = 11.2 \times 10^{-8} \text{ rad/ft}$$

$$\Delta\Psi = (\frac{\partial\Psi}{\partial\phi_{bo}}\Delta\phi_{bo}) + (\frac{\partial\Psi}{\partial v_{bo}}\Delta v_{bo}) + (\frac{\partial\Psi}{\partial r_{bo}}\Delta r_{bo})$$

$$\Delta\Psi = (-.566)\ (\frac{.2^\circ}{57.3^\circ/\text{rad}}) + (2.16 \times 10^{-4}\ \frac{\text{rad}}{\text{ft/sec}})\ (2\ \text{ft/sec}) + (11.2 \times 10^{-8}\ \text{rad/ft} \times 1000\ \text{ft})$$

$$\Delta\Psi = -.00198 + .000432 + .000112 = -.1436\ \text{rad}$$

$$\Delta\Psi = -.001436\ \text{rad} + 3440\ \frac{\text{NM}}{\text{rad}} = -4.94\ \text{NM}$$ *Ans.*

b. $$\Delta\Psi = (-30\ \frac{\text{NM}}{\text{deg}} \times .05\ \text{deg}) + 1.6\ \text{NM} + .41\ \text{NM} = -1.5\ \text{NM}$$

$$+ 1.6\ \text{NM} + .41\ \text{NM}$$

$$\Delta\Psi = .51\ \text{NM}$$ *Ans.*

c. $$\Delta\Psi = 0 = \left(\frac{\partial\Psi}{\partial\phi_{bo}}\Delta\phi_{bo}\right) + \left(\frac{\partial\Psi}{\partial v_{bo}}\Delta v_{bo}\right) + \left(\frac{\partial\Psi}{\partial r_{bo}}\Delta r_{bo}\right)$$

$$-\frac{\partial\Psi}{\partial v_{bo}}\Delta v_{bo} = -1.5\ \text{NM} + .41\ \text{NM} = -1.1\ \text{NM}$$

$$\Delta v_{bo} = \frac{-1.1\ \text{NM}}{-.8\ \frac{\text{NM}}{\text{ft/sec}}} = 1.4\ \frac{\text{ft}}{\text{sec}}$$ *Ans.*

Time of Flight of Ballistic Missiles

In choosing the trajectory for a ballistic missile the time of flight of the missile is an important consideration. The range from launch point to target on a rotating earth is affected by the time of flight. The time available for interception or other reaction depends upon the time of flight. The time of flight during free flight is of primary concern at this point.

The general time of flight methods are applicable to any ballistic missile trajectory, but there are certain cases in which the formulas and computations are particularly simple.

In Appendix D there is derived a formula for the time of flight of a ballistic missile for which the burnout and reentry heights are equal. It is given as:

$$t_\psi = 2\sqrt{\frac{a^3}{\mu}}\ (\pi - u_{bo} + \epsilon \sin u_{bo}) \qquad (16)$$

In Fig. 3 the time of flight on symmetrical trajectories vs the free-flight range angle is plotted for various values of $Q_{bo} = r_{bo}V^2_{bo}/\mu$ with $h_{bo} = 10^6$ ft. Note that for values of $Q_{bo} < 1$ there are two values of time of flight for each value of Ψ. The smaller value is for the low trajectory, the larger for the high trajectory. For $Q_{bo} > 1$ the time of flight is a single-valued function of the range since the distinction between high and low trajectory had disappeared.

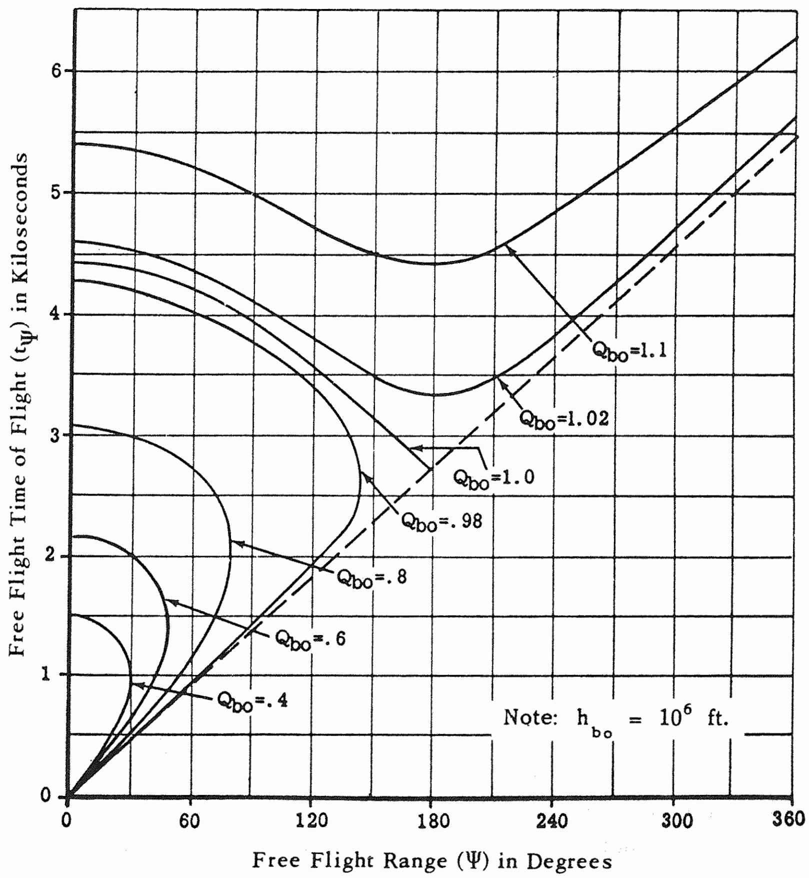


Figure 3. Free flight time of flight.

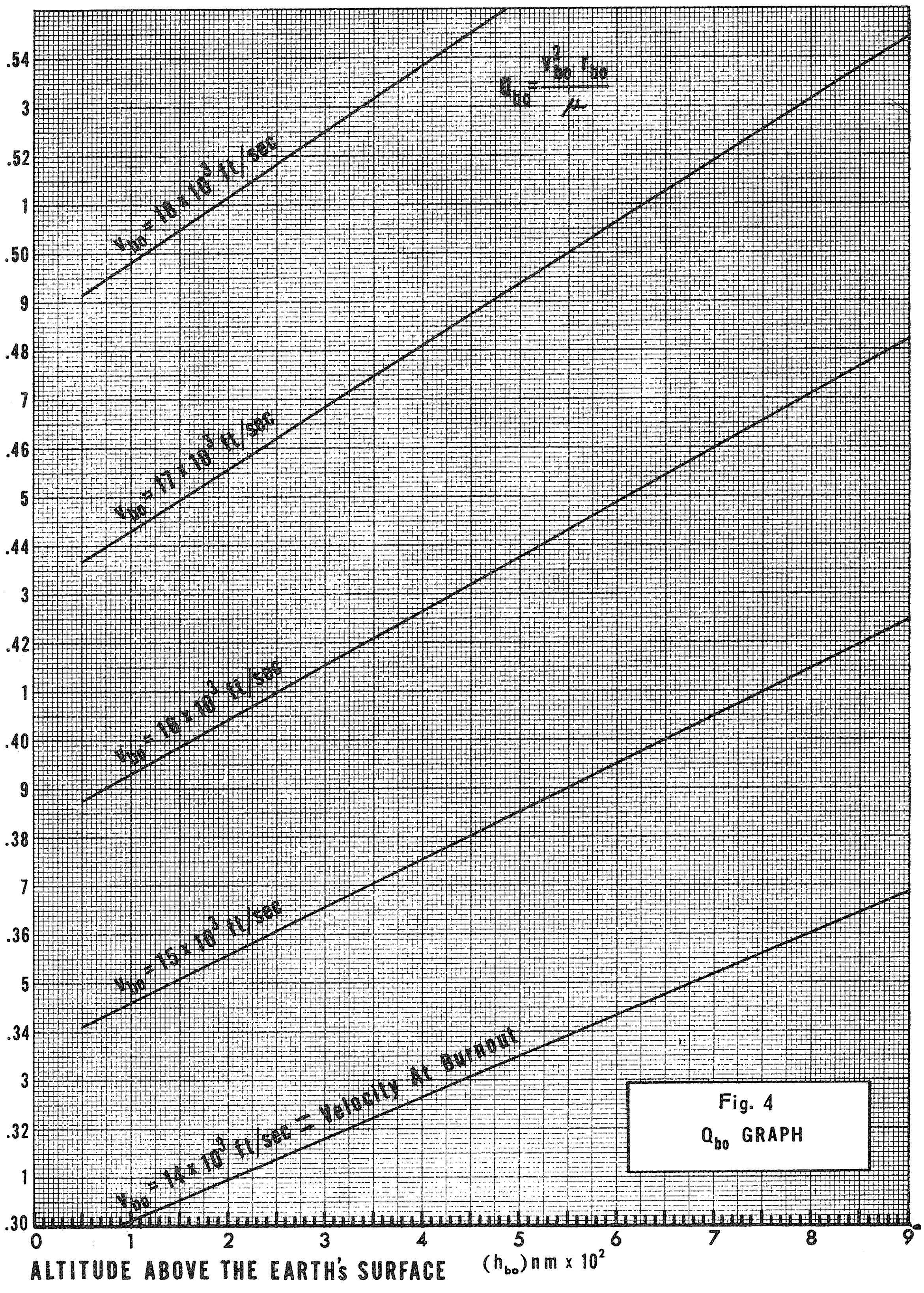


Fig. 4

Q_{bo} GRAPH

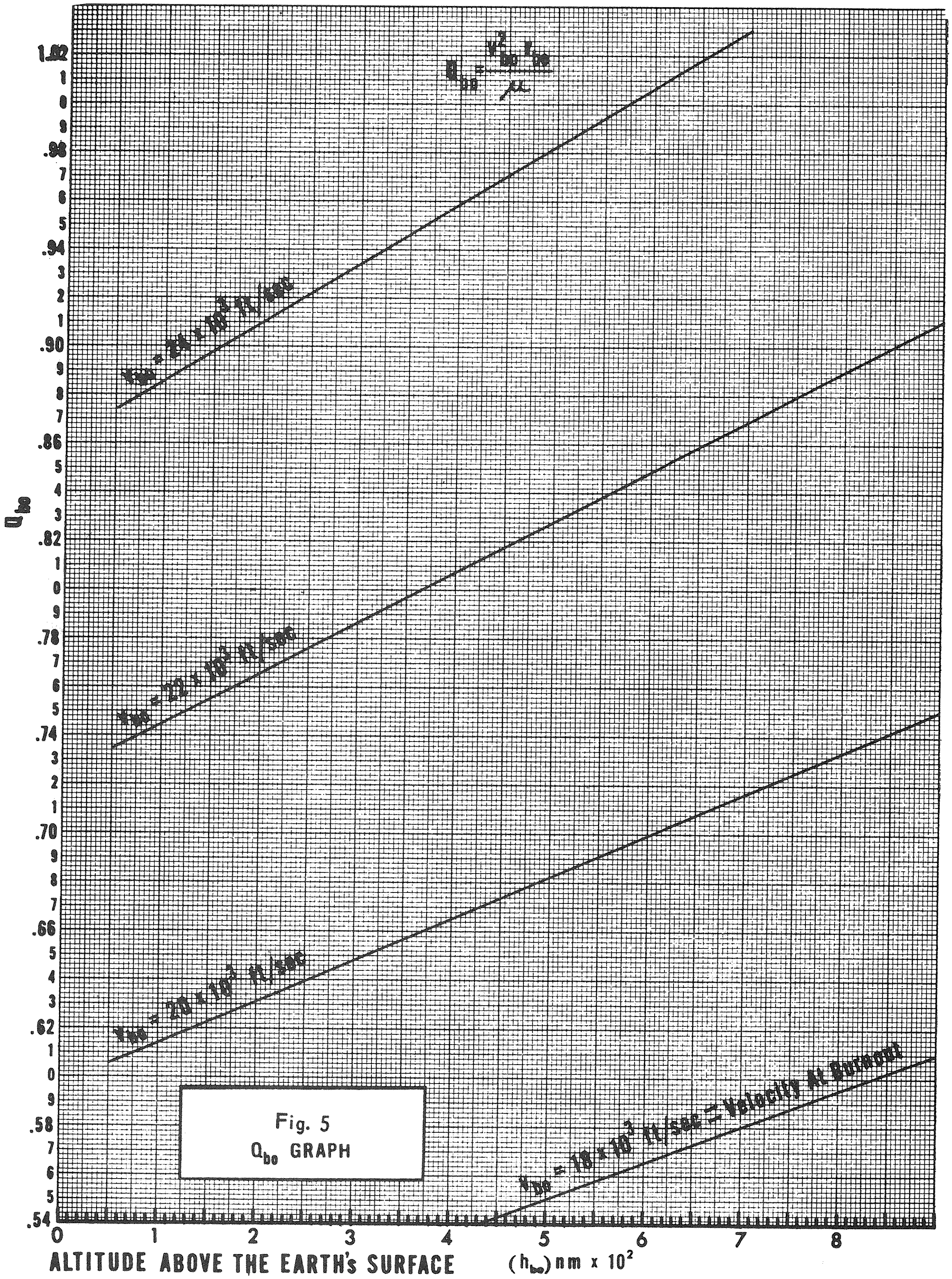


Fig. 5
Q_{bo} GRAPH

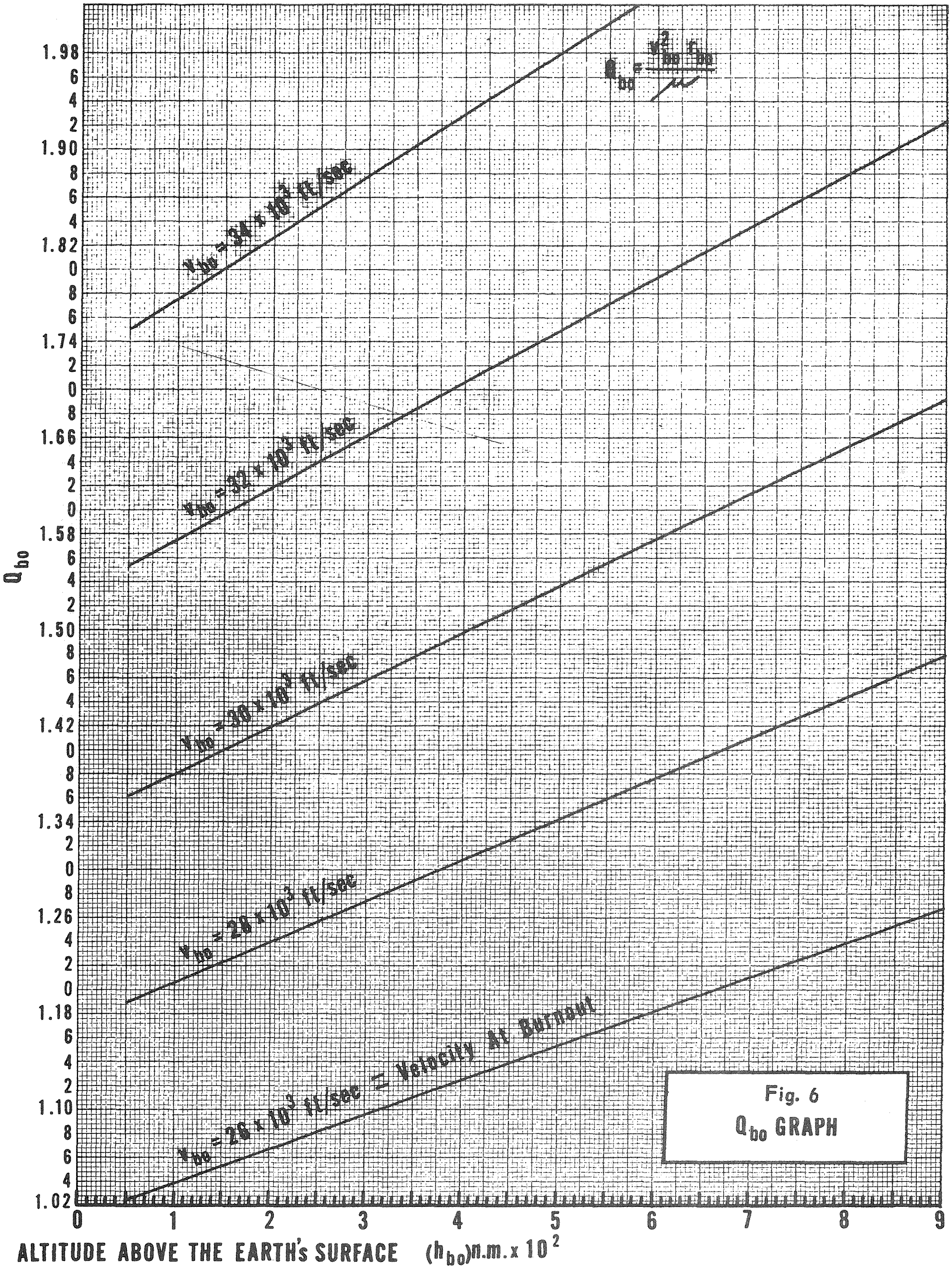


Fig. 6

Q_{bo} GRAPH

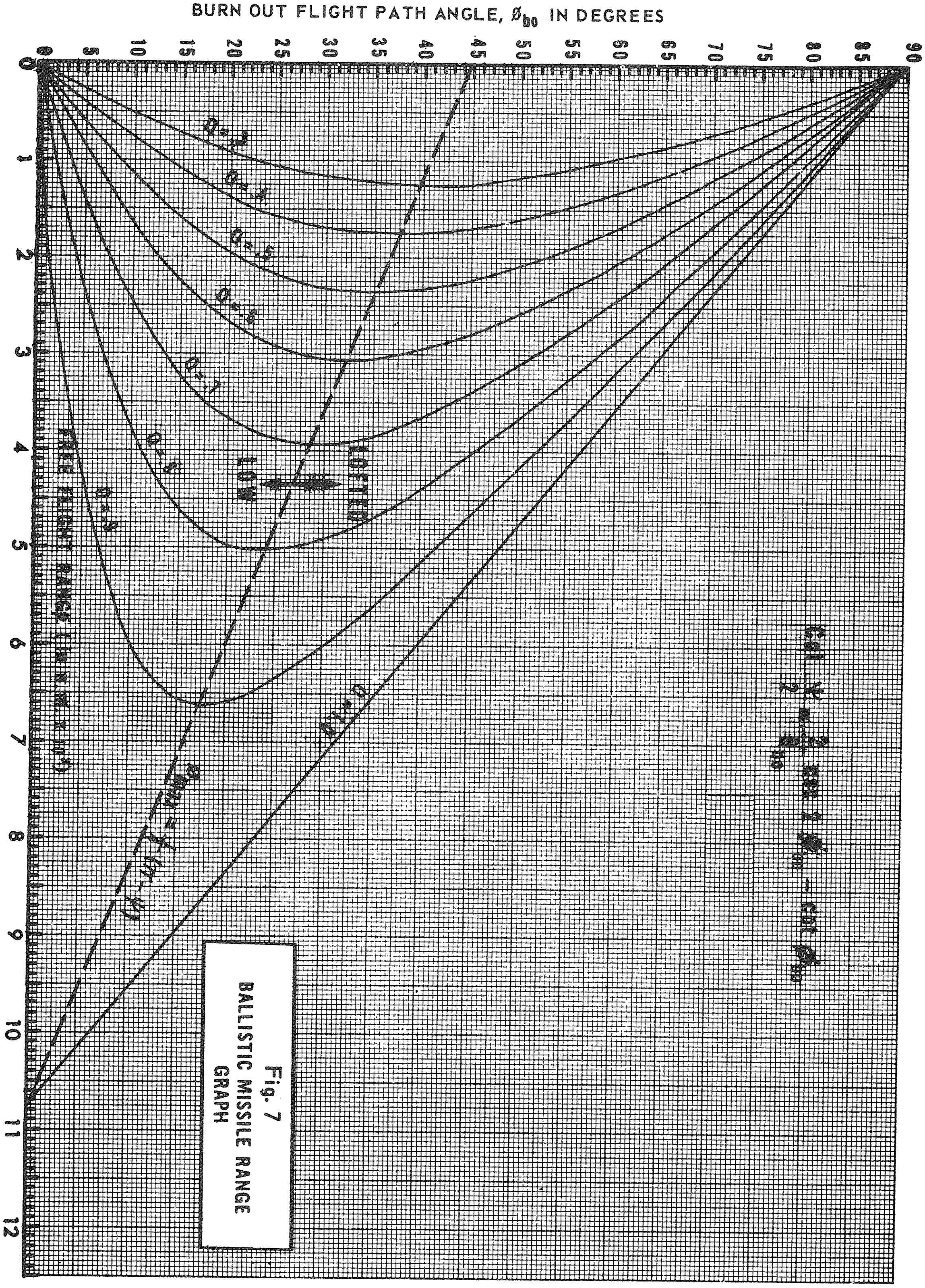


Fig. 7
BALLISTIC MISSILE RANGE GRAPH

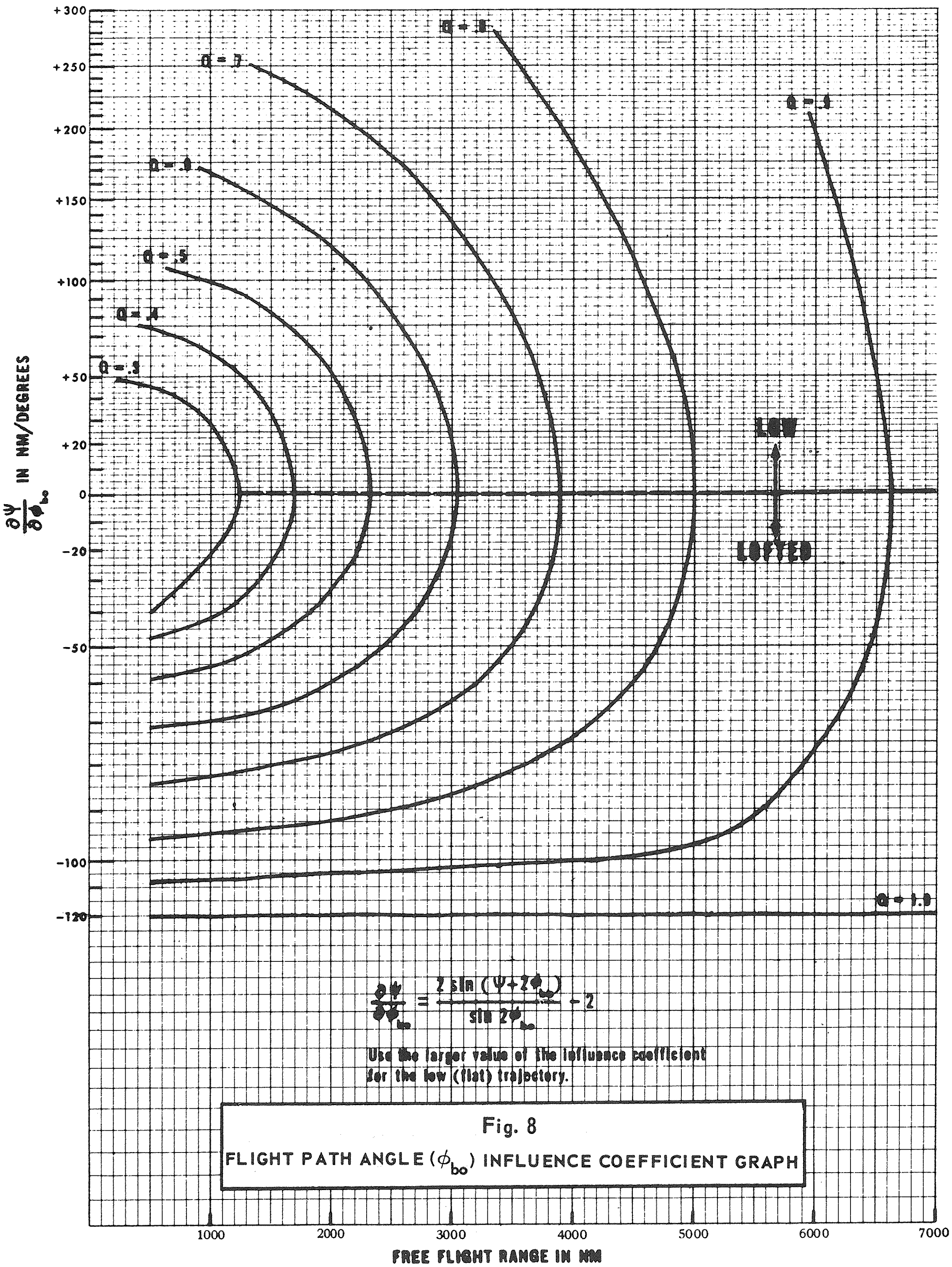


Fig. 8

FLIGHT PATH ANGLE (ϕ_{bo}) INFLUENCE COEFFICIENT GRAPH

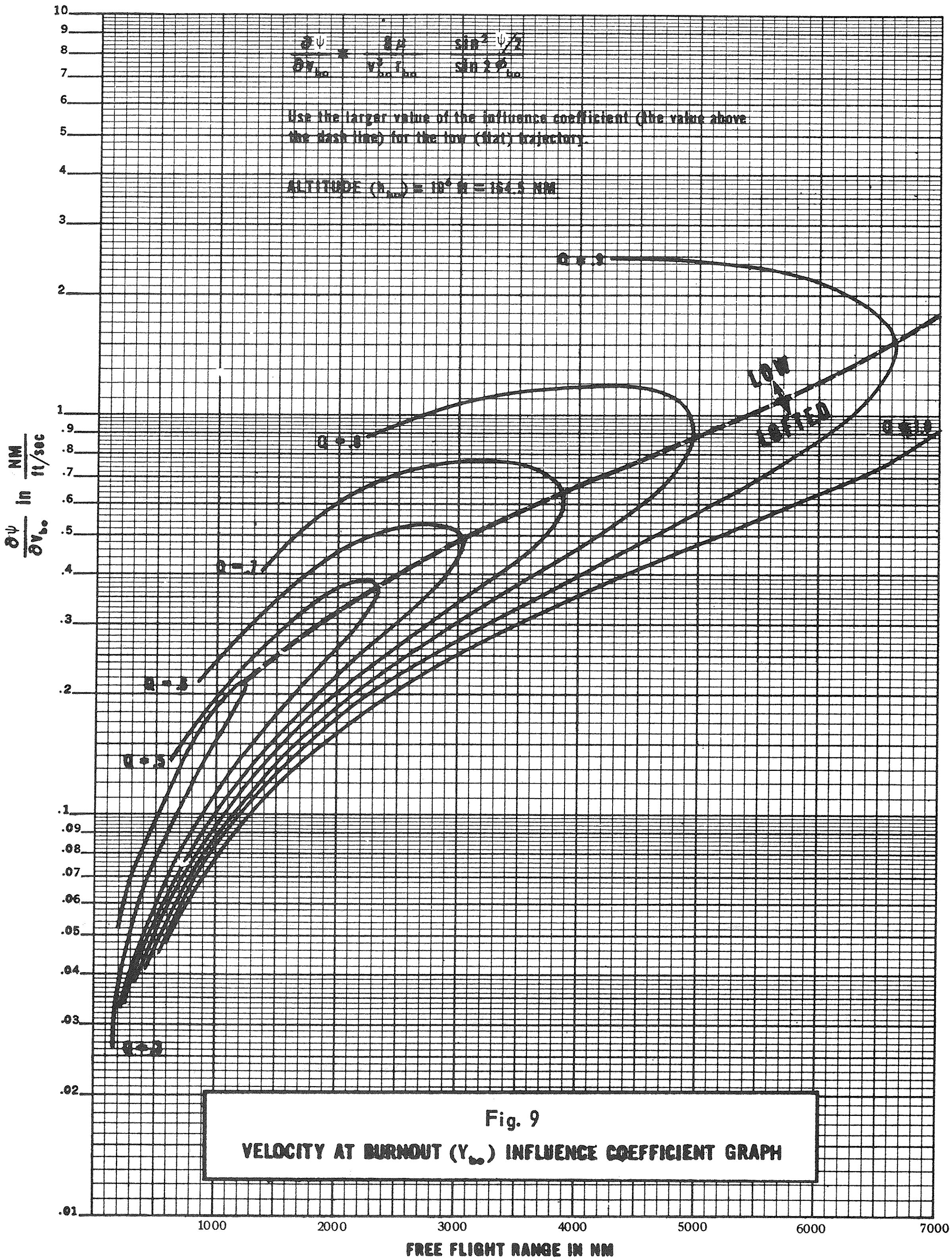


Fig. 9

VELOCITY AT BURNOUT (V_{bo}) INFLUENCE COEFFICIENT GRAPH

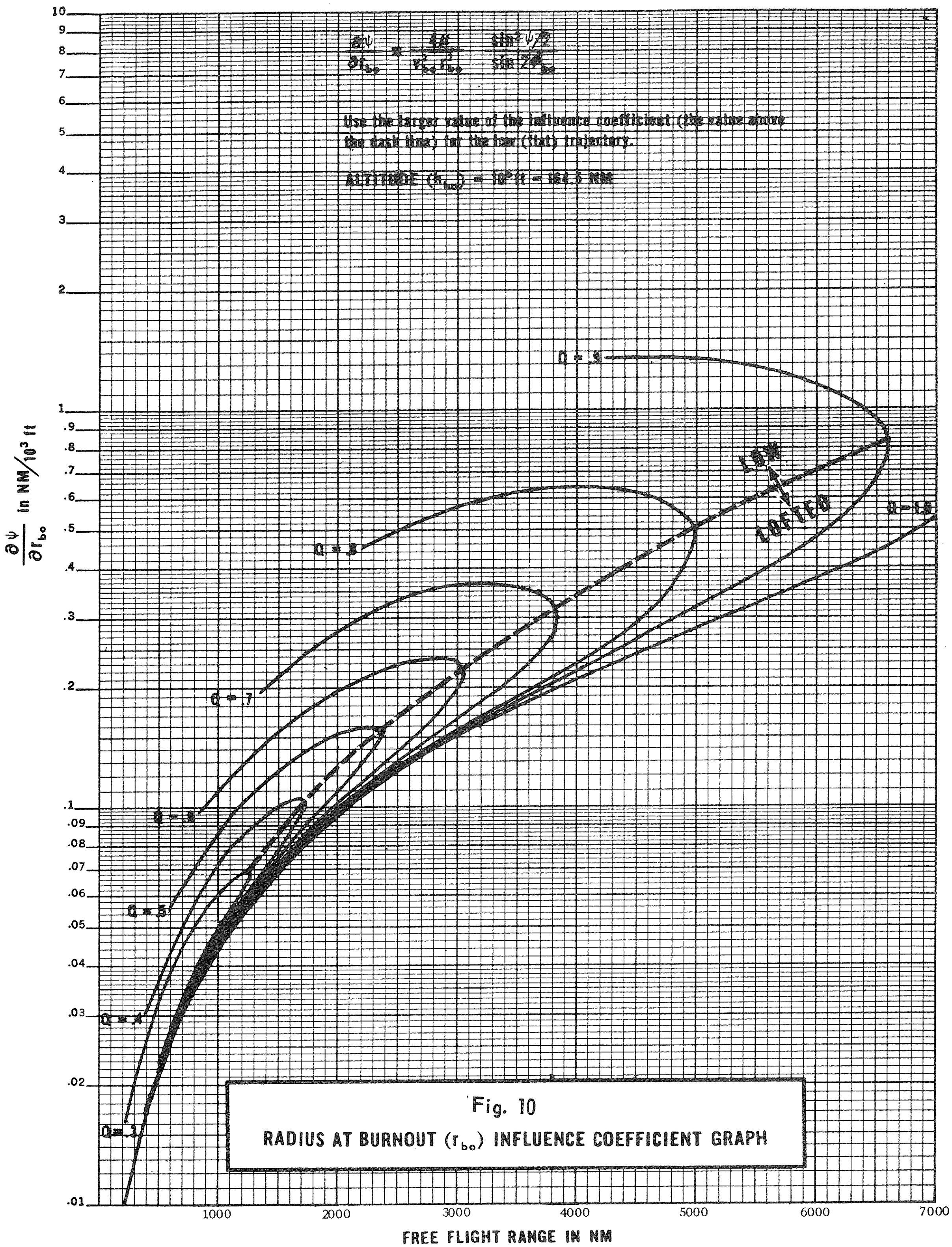


Fig. 10
RADIUS AT BURNOUT (r_{bo}) INFLUENCE COEFFICIENT GRAPH

Appendix D

TIME OF FLIGHT

THE TIME OF FLIGHT of an object traveling on an elliptical path is an important quantity. For example, the time of flight of a ballistic missile must be known in order to compute the apparent motion of the target due to the rotation of the earth. In order to derive an equation for the time of flight of a vehicle on an elliptical path, it is convenient to define three quantities known as anomalies.

In Fig. 1 an elliptical orbit with center of force at a focus F is shown. A circle of diameter equal to the major axis of the ellipse is drawn with center at the center of the ellipse, O. The anomalies of the general point R on the ellipse are defined as follows:

True Anomaly, ν—The angle BFR, measured from periapsis to the designated point on the flight path.

Eccentric Anomaly, u—A perpendicular to the major axis is dropped from R. This line intersects the circle at point S. The eccentric anomaly is the angle BOS. In the equations of this appendix, u is measured in radians.

Mean Anomaly, M—If t is the time of flight from the periapsis B to point R, and P is the period, the mean anomaly is:

$$M = 2\pi \frac{t}{P} \tag{1}$$

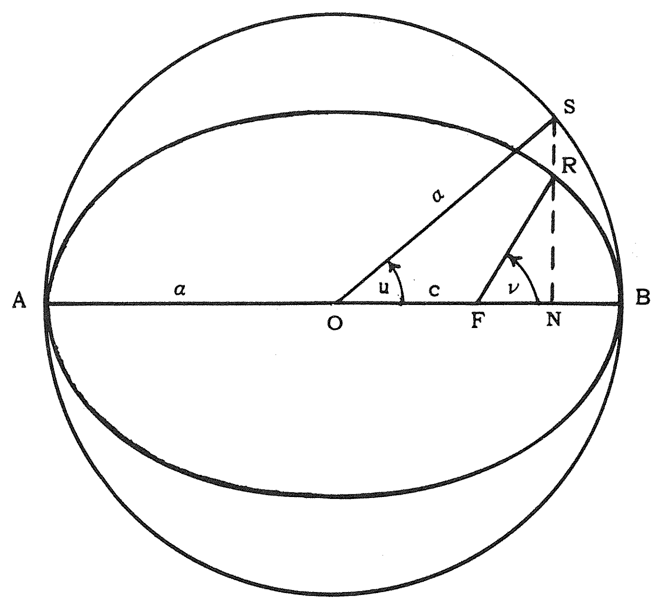


Figure 1. Geometry of the anomalies.

Note: All three anomalies are zero at the periapsis; all three are π at the apoapsis; if one is in the range O to π, the other two are also; and, if one is in the range π to 2π, the other two are also.

To obtain M, and hence t, apply Kepler's second law; area is swept out at a constant rate.

For the general point R, the area swept following periapsis passage is sector FBR. Then:

$$\frac{t}{P} = \frac{\text{Area FBR}}{\text{Area of ellipse}} \tag{2}$$

To determine these areas, write the equation of the ellipse and the auxiliary circle in rectangular coordinates, and compare the ordinates (y values) of the general point.

For the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ or } y_{\text{ellipse}} = \frac{b}{a}\sqrt{a^2 - x^2}$$

For the auxiliary circle of radius a:

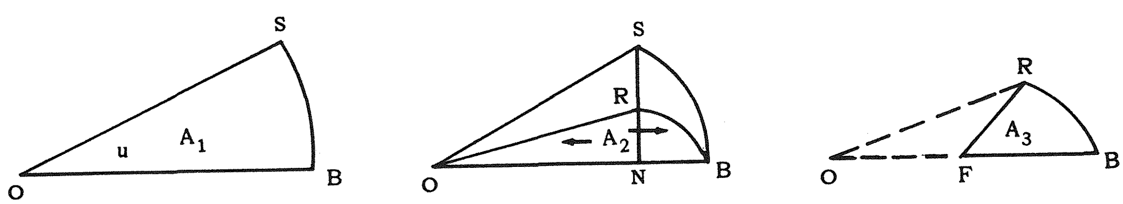
$$\frac{x^2}{a^2} + \frac{y^2}{a^2} = 1 \text{ or } y_{\text{circle}} = \sqrt{a^2 - x^2}$$

or

$$\frac{y_{\text{ellipse}}}{y_{\text{circle}}} = \frac{b}{a} \tag{3}$$

Equation (3) is important in subsequent comparisons of lengths and areas. For example, the area of the ellipse and the auxiliary circle in Figure 1 are related by:

$$\frac{\text{Area (ellipse)}}{\text{Area (circle)}} = \frac{\pi\, ab}{\pi a^2} = \frac{b}{a}$$



Examine the following areas:

In Figure 2(a) the area of the circular sector is the fraction $\left(\frac{\text{u radians}}{2\pi \text{ radians}}\right)$ of the area (π a^2) of the auxiliary circle; therefore:

$$A_1 = \frac{u}{2\pi}(\pi\, a^2) = \frac{u\, a^2}{2}$$

In Figure 2(b) triangles ONR and ONS have the same base, and their heights (therefore areas) are in the ratio $\frac{NR}{NS} = \frac{b}{a}$. The areas of the elliptical segment NBR and circular segment NBS are in the same ratio. Therefore:

$$A_2 = \frac{b}{a} A_1 = \frac{b}{a}\left(\frac{u\,a^2}{2}\right) = \frac{u\,ab}{2}$$

The area (A_3) swept following periapsis passage is obtained by subtracting the triangle OFR from A_2. The base of the triangle OFR is $c = a\epsilon$, and, in terms of u, its height is $\frac{b}{a}$ (a sin u) = b sin u.

Then:

$$A_3 = A_2 - \tfrac{1}{2}(a\epsilon)(b \sin u) = \frac{u\,ab}{2} - \frac{\epsilon\,ab}{2} \sin u = \frac{ab}{2}(u - \epsilon \sin u)$$

From Equation (2):

$$\frac{t}{P} = \frac{A_3}{\pi\,ab} = \frac{\frac{ab}{2}(u - \epsilon \sin u)}{\pi\,ab} = \frac{1}{2\pi}(u - \epsilon \sin u)$$

$$\text{or } t = \frac{P}{2\pi}(u - \epsilon \sin u)$$

$$\text{Where } P = 2\pi\sqrt{\frac{a^3}{\mu}} \tag{4}$$

From Equation (1)

$$M = 2\pi \frac{t}{P} = u - \epsilon \sin u \tag{5}$$

Then the time from periapsis to any point on an elliptical orbit can be found from the definition of the mean anomaly and the known period.

$$t = \frac{P}{2\pi}(u - \epsilon \sin u) = \sqrt{\frac{a^3}{\mu}}(u - \epsilon \sin u) \tag{6}$$

The time of flight between any two points in general (1 and 2) depends upon the difference in the mean anomalies of the points:

$$t_{1\to 2} = \frac{P}{2\pi}(M_2 - M_1) = \sqrt{\frac{a^3}{\mu}}(M_2 - M_1)$$

$$t_{1\to 2} = \sqrt{\frac{a^3}{\mu}}\,[(u_2 - \epsilon \sin u_2) - (u_1 - \epsilon \sin u_1)] \tag{7}$$

Note: If u_1 is greater than u_2, periapsis is passed in the transit time, and Equation (6) results in a negative value. This value is the difference between the correct time of flight and one period; to eliminate the problem of negative values, replace u_2 with $2\pi + u_2$.

For most practical problems it is necessary to relate the eccentric anomaly u to the true anomaly ν.
From Figure 1:

$$\cos u = \frac{c + r\cos\nu}{a}$$

$$\text{Since } c = a\epsilon \text{ and } r = \frac{a(1-\epsilon^2)}{1+\epsilon\cos\nu}$$

$$\cos u = \frac{a\epsilon + \dfrac{a(1-\epsilon^2)\cos\nu}{1+\epsilon\cos\nu}}{a}$$

$$\cos u = \frac{\epsilon(1+\epsilon\cos\nu) + (1-\epsilon^2)\cos\nu}{1+\epsilon\cos\nu}$$

$$\cos u = \frac{\epsilon + \epsilon^2\cos\nu + \cos\nu - \epsilon^2\cos\nu}{1+\epsilon\cos\nu}$$

$$\cos u = \frac{\epsilon + \cos\nu}{1+\epsilon\cos\nu} \tag{8}$$

Similarly:

$$\sin u = \frac{\sqrt{1-\epsilon^2}\,\sin\nu}{1+\epsilon\cos\nu} \tag{9}$$

The path of a ballistic missile (Fig. 3) is nearly symmetrical from burnout to reentry.

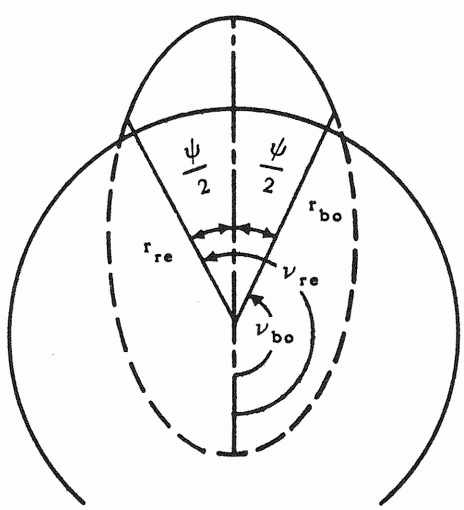


Figure 3. Geometry of ballistic missile time of flight.

If the range angle is ψ, the following relationships are noted:

$$\nu_{bo} = \pi - \frac{\psi}{2} \qquad \cos \nu_{bo} = -\cos \frac{\psi}{2} \qquad \sin \nu_{bo} = \sin \frac{\psi}{2}$$

$$\nu_{re} = \pi + \frac{\psi}{2} \qquad \cos \nu_{re} = -\cos \frac{\psi}{2} \qquad \sin \nu_{re} = -\sin \frac{\psi}{2}$$

$$\nu_{re} = 2\pi - \nu_{bo} \qquad \cos \nu_{re} = \cos \nu_{bo} \qquad \sin \nu_{re} = -\sin \nu_{bo}$$

Assuming 1 to be the burnout point and 2 to be the reentry point, Equation (7) becomes:

$$t_{bo \to re} = t_{\psi} = \sqrt{\frac{a^3}{\mu}} \left[(2\pi - u_{bo} + \epsilon \sin u_{bo}) - (u_{bo} - \epsilon \sin u_{bo})\right]$$

$$t_{\psi} = \sqrt{\frac{a^3}{\mu}} (2\pi - 2u_{bo} + 2\,\epsilon \sin u_{bo})$$

$$t_{\psi} = \sqrt{\frac{a^3}{\mu}} (\pi - u_{bo} + \epsilon \sin u_{bo}) \tag{10}$$

From Equation (8):

$$\cos u_{ob} = \frac{\epsilon + \cos \nu_{bo}}{1 + \epsilon \cos \nu_{bo}} = \frac{\epsilon - \cos \frac{\psi}{2}}{1 - \epsilon \cos \frac{\psi}{2}} \tag{11}$$

Appendix E

PROPULSION

Discussion of Vehicle Net Acceleration at Launch

THE ACCELERATION of a launch vehicle as it leaves the pad is caused by the difference between the engine thrust and the vehicle weight. From Newton's second law:

$$f = Ma$$

or

$$a = \frac{f}{M}$$

The net force (f) in this case is vehicle thrust (F) minus vehicle weight (W) at lift-off, and M is the mass of the object acted upon by the force differential, i.e., the vehicle itself. Then:

$$a = \frac{F-W}{M}$$

$$\text{but } M = \frac{W}{g}$$

$$\therefore a = \left(\frac{F-W}{W}\right) g$$

or

$$a = \left(\frac{F}{W} - \frac{W}{W}\right) g = (\Psi - 1)\ g$$

where Ψ is the thrust-to-weight ratio.

$\therefore a = (\Psi - 1)$ g's (Equation 6, Chapter 3)

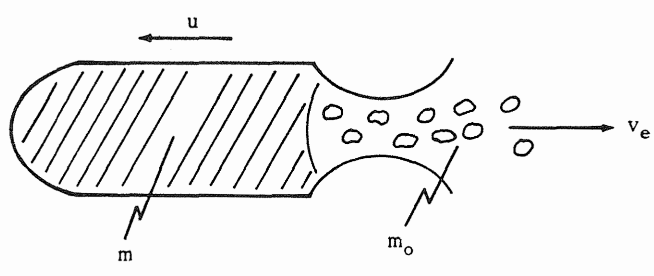


Figure 1

Discussion of Rocket Engine Thrust Equation

The efflux of hot gas from a rocket can be regarded as the ejection of small masses such as m_o (gas molecules) at a high relative velocity v_e with respect to the vehicle, which has mass m and is moving at velocity u. According to Newton's second law, the sum of the unbalanced forces acting on the vehicle is equal to the time rate of change of the vehicle's momentum. The system being considered is the overall vehicle. With these ideas in mind:

$$\Sigma f = \frac{d}{dt}(mv) = m\frac{dv}{dt} + v\frac{dm}{dt}$$

The m is the instantaneous vehicle mass. $\frac{dv}{dt}$ applies to the vehicle and should be $\frac{du}{dt}$. v in the second term applies to the ejected masses which is v_e. And, $\frac{dm}{dt}$ applies to either vehicle or gases since $-\frac{dm}{dt} = \frac{dm_o}{dt}$. Using $-\frac{dm}{dt}$ yields:

$$\Sigma f = m\frac{du}{dt} - v_e\frac{dm}{dt}$$

If the vehicle is operating in a "weightless" environment free of an atmosphere and under steady state conditions, the sum of the unbalanced forces equals 0. Thus:

$$\Sigma f = 0 = m\frac{du}{dt} - v_e\frac{dm}{dt}$$

and

$$m\frac{du}{dt} = v_e\frac{dm}{dt}$$

The left side of the equation represents the constant (steady state) unbalanced force (thrust) on the vehicle ($F = m\frac{du}{dt}$ from Newton's second law). Then:

$$\text{Thrust} = F = v_e\frac{dm}{dt}$$

Since F is constant for steady state, the variables may be separated and integration applied, or one may recognize $\frac{dm}{dt}$ as the mass rate of flow of the rocket exhaust (the propellants). Either way, for an atmosphere-free environment, momentum thrust is:

$$F = \dot{M}v_e$$

or

$$F = \frac{\dot{W}}{g}v_e$$

When the vehicle flies through an ambient fluid such as the earth's atmosphere, the flow around it is affected, and the fluid can interact with the rocket exhaust. Under these conditions the thrust relationships must be corrected by the effect of the pressure forces which act on the surfaces of the vehicle body. The figure below shows the ambient pressure acting uniformly on the outer surface of a rocket chamber and the gas pressures on the inner surface.

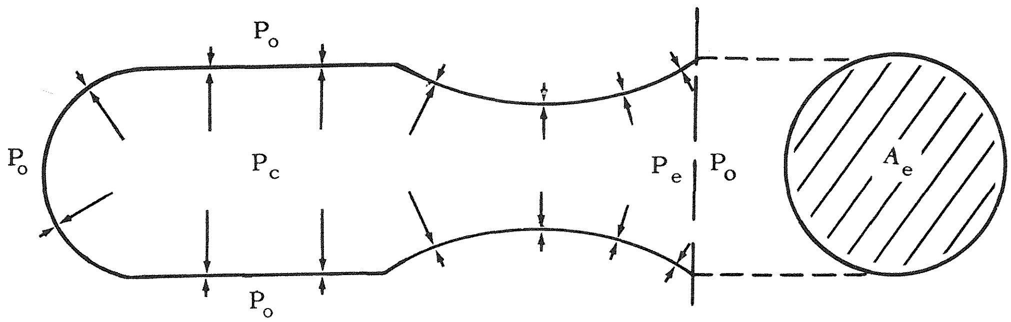


Figure 2

The size of the arrows indicates the relative magnitude of the pressure. The axial thrust can be determined by integrating all the pressure acting on areas which can be projected on a plane normal to the nozzle axis. The forces acting radially outward are appreciable but balance each other and do not contribute to axial thrust.

If the plane upon which all integrated pressures are projected is that of the engine exhaust, one finds that at the exit area A_e there is an imbalance between ambient (P_o) and the local pressure of the exhaust (P_e), the difference being ($P_e - P_o$). The differential force involved is evaluated over the area of the nozzle exit and is equal to $A_e(P_e - P_o)$.

The total thrust acting on the rocket then, must be the sum of the momentum and pressure thrusts or:

$$F = \frac{\dot{W}}{g} v_e + A_e (P_e - P_o) \text{ [Equation 1, Chapter 3]}$$

The second term on the right side of this equation (pressure thrust) affects thrust in this way. If the exhaust pressure is less than ambient pressure, the pressure thrust is negative. Because this condition gives low thrust, the nozzle is usually designed so that P_e equals or slightly exceeds P_o. When $P_e = P_o$ maximum thrust for a given propellant and chamber is attained.

Discussion of Ideal Velocity of a Rocket at Thrust Termination

The equation for the magnitude of the "ideal" vehicle velocity at thrust termination is $\Delta v_i = I_{sp}\, g \ln \left(\frac{W_1}{W_2}\right)$. It can be derived from Newton's second law using the following assumptions:

1. Propellant used at a constant rate.

2. Thrust constant.

Newton's second law may be stated in *any consistent units.*

$$F = Ma = f \text{ (no gravity; no drag)} \tag{A}$$

or

$$F = M \frac{dv}{dt} \tag{B}$$

Since propellant is used at a constant rate the mass of the vehicle at any time is:

$$M = M_1 - \dot{M}t \tag{C}$$

Substitute equation (C) in equation (B):

$$F = (M_1 - \dot{M}t) \frac{dv}{dt} \tag{D}$$

Separate variables and integrate:

$$\int_{v_1}^{v_2} dv = \int_{o}^{t_2} \frac{Fdt}{(M_1 - \dot{M}t)} \tag{E}$$

Since thrust is constant:

$$v_2 - v_1 = F \int_{t_i}^{t_2} \frac{dt}{(M_1 - \dot{M}t)} \tag{F}$$

Since propellant is used at a constant rate:

$$v_2 - v_1 = - \frac{F}{\dot{M}} \Big[\ln (M_1 - \dot{M}t) \Big]_{o}^{t_2} \tag{G}$$

When $t_i = o$:

$$M_1 - \dot{M}(o) = M_1 \tag{H}$$

When $t = t_2$:

$$M_1 - \dot{M}t_2 = M_2 \tag{I}$$

Substitute equations (H) and (I) in equation (G):

$$v_2 - v_1 = - \left[\frac{F}{\dot{M}} \right] \Big[\ln M_2 - \ln M_1 \Big] \tag{J}$$

Note: $-(\ln M_2 - \ln M_1) = (\ln M_1 - \ln M_2)$

$$v_2 - v_1 = \Delta v = \frac{F}{\dot{M}} \ln\left(\frac{M_1}{M_2}\right) \qquad \text{(K)}$$

If we use the English Engineering System, mass is measured in slugs (M) and weight is measured in pounds (W).

$$\frac{M_1}{M_2} = \frac{M_1 g}{M_2 g} = \frac{W_1}{W_2} \qquad \text{(L)}$$

$$\dot{M} = \frac{\dot{W}}{g} \quad \textit{Converting mass to weight at earth's surface.} \qquad \text{(M)}$$

$$I_{sp} = \frac{F}{\dot{W}} \qquad \text{(N)}$$

Substitute equations (L) (M) and (N) in equation (K) and obtain:

$$\Delta v_i = I_{sp}\, g \ln\left(\frac{W_1}{W_2}\right) \text{ (Equation 7, Chapter 3)} \qquad \text{(O)}$$

Discussion of Mass Ratio for Multi-Stage Rockets

Equation (O) from the previous section can now be applied to a multistage launch vehicle if it is assumed that the velocity of each stage has the same direction. Then the magnitude of the vehicle velocity at thrust termination of the last stage is the sum of the velocity magnitudes of each stage. Consider a three stage vehicle with *each stage having the same specific impulse.*

Stage 1: $\Delta v_1 = I_{sp}\ g \ln\left(\frac{W_1}{W_2}\right)_1$

Stage 2: $\Delta v_2 = I_{sp}\ g \ln\left(\frac{W_1}{W_2}\right)_2$

Stage 3: $\Delta v_3 = I_{sp}\ g \ln\left(\frac{W_1}{W_2}\right)_3$

$$\Delta v_i = \Delta v_1 + \Delta v_2 + \Delta v_3 = I_{sp}\,(g)\left[\ln\left(\frac{W_1}{W_2}\right)_1 + \ln\left(\frac{W_1}{W_2}\right)_2 + \ln\left(\frac{W_1}{W_2}\right)_3\right]$$

This equation contains the sum of three natural logarithms in the brackets. The sum of the natural logarithms of several numbers is the logarithm of the product of the numbers. Therefore, the quantity in the brackets is the natural logarithm of the product of the stage mass ratios.

$$\Delta v_i = I_{sp}\, g \ln\left(\frac{W_1}{W_2}\right)_1\left(\frac{W_1}{W_2}\right)_2\left(\frac{W_1}{W_2}\right)_3$$

This is the basis for multiplying the individual mass ratios of each stage to get the overall mass ratio of a multistage vehicle.

Discussion of Electric Rocket Thrust Equation

The electric power input and power output of an electric propulsion system can be equated if a parameter called efficiency is placed in the equation. Thus:

(Power in) (efficiency) = power out

In terms of electrical power and kinetic energy per unit time, this equation may be written:

$$\eta p = \frac{\dot{M} v_e^2}{2}$$

where η = efficiency (%)

$\dot{M}$ = mass flow rate of expelled particles $\left(\frac{\text{slugs}}{\text{sec}}\right)$

v_e = exhaust velocity of expelled particles (ft/sec)

p = electric power input (kw)

From the derivation of the rocket thrust equation, momentum thrust (F) is expressed as $\dot{M} v_e$ (there is no appreciable pressure thrust in an electric engine).

$$\text{then } \eta p = \frac{(\dot{M} v_e) v_e}{2} = \frac{F v_e}{2}$$

$$\text{but } F = \dot{M} v_e \text{ or } v_e = \frac{F}{\dot{M}}$$

$$\text{then } \eta p = \frac{(F)\,(F)}{2\dot{M}}$$

$$\text{and } F^2 = 2\eta p \dot{M}$$

To make units compatible on both sides of this equation, a conversion constant ($K = 737.56 \frac{\text{ft lb}}{\text{kw sec}}$) must be applied.

This yields:

$$F^2 = 2 K \eta p \dot{M}$$

$$F = \sqrt{2(737.56)\eta p \dot{M}}$$

$$F = 38.4 \sqrt{\eta p \dot{M}}$$ (Equation 16 at end of Chapter 3)

Discussion of Theoretical Specific Impulse

The principle of conservation of energy may be written for an isentropic process between any two points in a rocket engine. In this discussion the two points under consideration are the combustion chamber and nozzle exit. Begin with the energy equation in which the decrease in specific enthalpy is equal to the increase in kinetic energy of the flowing gases.

$$\therefore h_c - h_e = \frac{1}{2gJ}(v_e^2 - v_c^2)$$

where: h_c = specific enthalpy in combustion chamber
h_e = specific enthalpy at nozzle exit
g = 32.2 ft/sec^2
J = mechanical equivalent of heat (778 ft-lb/BTU)
v_e = velocity of exhaust gases
v_c = velocity of gases in the combustion chamber

Since $v_c \approx O$ in the longitudinal direction due to the random motion of gases in the combustion chamber, neglect its contribution and rearrange the equation to give:

$$v_e^2 = 2g\,J\Delta h \qquad \text{where } \Delta h = h_c - h_e$$

Also, in the isentropic flow of a perfect gas:

$$\Delta h = C_p \Delta T$$

where: C_p = specific heat of a gas at constant pressure
$\Delta T = T_c - T_e$
T_c = temperature of gas in combustion chamber
T_e = temperature of exhaust gases

$$\therefore v_e^2 = 2g\,J\,C_p\,(T_c - T_e)$$

and since $C_p = \left(\frac{k}{k-1}\right)\left(\frac{R}{J}\right)$

where: k = ratio of specific heats

$$R = \frac{R'}{m} = \frac{\text{universal gas constant}}{\text{molecular weight of combustion products}}$$

$$\therefore v_e^2 = 2gR'\left(\frac{k}{k-1}\right)\frac{T_c}{m}\left(1 - \frac{T_e}{T_c}\right)$$

In the isentropic process of a perfect gas:

$$\frac{T_e}{T_c} = \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}$$

$$\therefore v_e^2 = 2g\,\frac{k}{k-1}\,R'\,\frac{T_c}{m}\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]$$

and since R' = 1544 ft-lb/°R mole:

$$v_e^2 = 2\,(32.2)(1544)\ \frac{k}{k-1}\ \frac{T_c}{m}\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]$$

$$\text{or } v_e = 315.1\ \sqrt{\frac{k}{k-1}\ \frac{T_c}{m}\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]}$$

and since:

$$F = \frac{\dot{W}}{g}\ v_e \text{ (optimum expansion)}$$

$$\frac{F}{\dot{W}} = I_{sp} = \frac{v_e}{g}$$

$$\therefore\ I_{sp} = \frac{315.1}{32.2}\sqrt{\frac{k}{k-1}\ \frac{T_c}{m}\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]}$$

$$I_{sp} = 9.797\ \sqrt{\frac{k}{k-1}\ \frac{T_c}{m}\left[1 - \left(\frac{P_e}{P_c}\right)^{\frac{k-1}{k}}\right]} \quad \text{(Equation 9 in Chapter 3)}$$

Discussion of Velocity Variations Contributing to Actual Vehicle Velocity

In the equation:

$v_{ACT} = \Delta v(\text{Ideal}) + v_r - \Delta v_{loss}$ (Equation 8, Chapter 3)

v_{ACT} = magnitude of vehicle velocity at last stage thrust termination

v_r = component of earth's tangential velocity in the direction of the launch azimuth

Δv_{loss} = magnitude of velocity loss due to drag and gravity

The contribution v_r can be calculated from launch azimuth and latitude by the equation:

$$v_r = r_e \omega_e \cos L \sin \beta$$

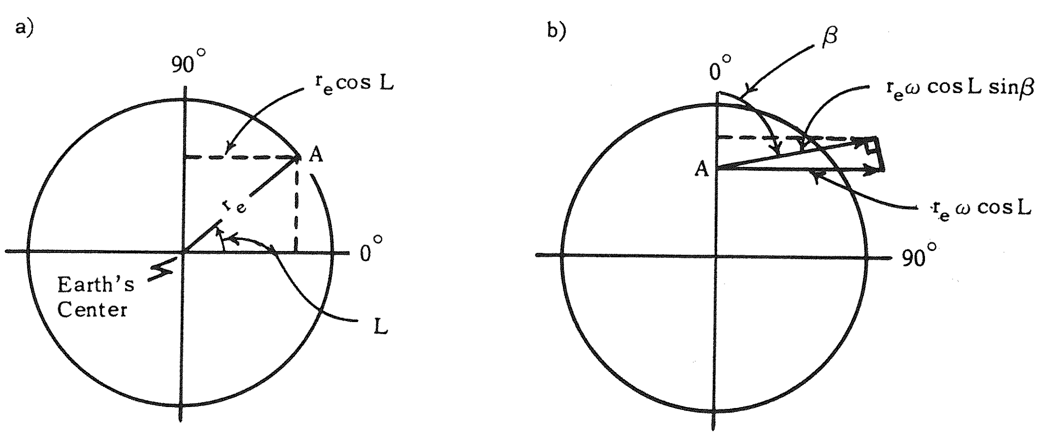


Figure 3

Where: r_e = earth radius (20.9×10^6 feet)

ω_e = earth angular rotation (7.27×10^{-5} radians/sec)

L = launch latitude

β = launch azimuth

In figure 3 a) the effect of launch latitude is illustrated.

In figure 3 b) the effect of launch azimuth is illustrated.
Therefore:

$$v_r = (20.9 \times 10^6)\ (7.27 \times 10^{-5}) \cos L \sin \beta$$

$$v_r = 1520 \cos L \sin \beta$$

Notice that the maximum contribution to the actual velocity by the effect of the earth's rotation is for an easterly launch at the equator ($L = 0°$) ($\beta = 90°$).

The term Δv_{loss} is calculated from:

$$\Delta v_{loss} = \int g \sin \phi \, dt + \int \frac{\rho v^2 C_d A \, dt}{2M}$$

Where: g = gravitational force (ft/sec²)

ϕ = angle between flight path and perpendicular to radius vector at any altitude (degrees)

dt = time increment (sec)

ρ = atmospheric density (slugs/ft³)

v = vehicle velocity (ft/sec)

C_d = coefficient of drag

A = effective vehicle area (ft²)

M = vehicle mass (slugs)

Calculation of Δv_{loss} is a complex iterative process. A good approximation of Δv_{loss} is 5000 ft/sec for present launch vehicles.
Data for an actual Titan II launch provides a good example of Δv_{loss}:

v_r = 1100 ft/sec

Δv_1 = 12,500 ft/sec

Δv_2 = 16,900 ft/sec

Δv (Ideal) = 29,400 ft/sec

but Δv_{ACT} = 25,756 ft/sec

$\therefore\ \Delta v_{loss}$ = 4,744 ft/sec

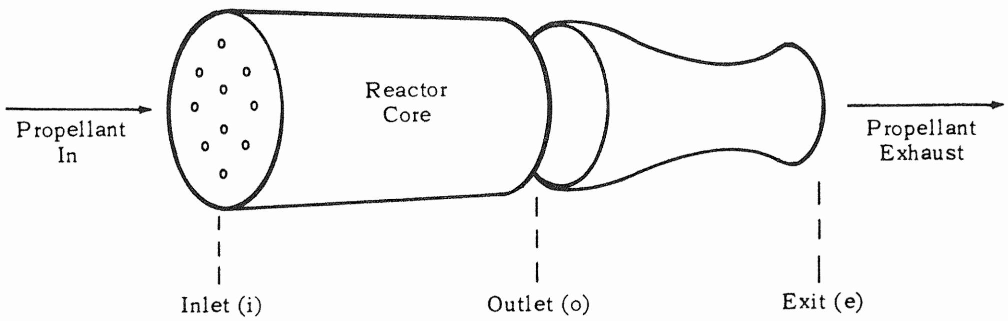


Figure 4

Discussion of Nuclear Rocket Thrust Equation

In the discussion of theoretical specific impulse which occurs earlier in this appendix, the decrease in enthalpy of the exhaust gases flowing through the nozzle of a chemical rocket engine was set equal to the increase in kinetic energy of the flowing gases. This relationship is true also for the propellant flow through the nozzle of a nuclear rocket such as the NERVA engine as shown below.

$$h_o - h_e = \frac{1}{2gJ}\left(v_e^2 - v_o^2\right) \qquad \text{(A)}$$

where: h = specific enthalpy (BTU/lb)

g = 32.2 ft/sec^2

v = velocity (ft/sec)

J = mechanical equivalent of heat ($778 \frac{\text{ft lb}}{\text{BTU}}$)

If Q represents total reactor thermal power (BTU/sec) generation, then $Q/\dot{W} = q$ represents the specific input of heat (BTU/lb) available for transfer to the propellant when $\dot{W}$ is the propellant flow rate (lb/sec).

Because the propellant will not receive all the reactor heat generated (losses and inefficiency in heat transfer), the change in the propellant specific enthalpy across the core will be:

$$h_o - h_i = \nu q$$

where: ν = fraction of q delivered to the propellant

then: $h_o = \nu q + h_i$ (B)

Substituting h_o from equation (B) into equation (A) yields:

$$\nu q + h_i - h_e = \frac{1}{2gJ}\left(v_e^2 - v_o^2\right) \qquad \text{(C)}$$

v_o^2 is very small with respect to v_e^2 and may be neglected. And the heat addition and expansion processes occur at nearly a constant pressure so that:

$$h_i - h_e = \int_e^i C_p \, dT \tag{D}$$

where: C_p = propellant constant pressure specific heat $\left(\frac{\text{BTU}}{\text{lb}\ ^\circ\text{R}}\right)$

T = propellant bulk (average) temperature (°R)

It appears that the propellant to be used in nuclear rockets in the foreseeable future will be hydrogen. The remainder of this discussion will assume H_2 to be the propellant.

For gaseous H_2, C_p varies between 3.4 and 5.0 $\frac{\text{BTU}}{\text{lb}\ ^\circ\text{R}}$ over a temperature range of 600 to 5000° R. This range of variation could be overcome by approximating C_p for hydrogen by $C_p = 2.857 + 2.867\ (10^{-4})\ T + 9.92T^{-1/2}$ (See pg 61 of *Thermodynamics* by Faires, listed in the Bibliography at the end of Chapter 3 of this text, for a fuller discussion.) This could be used in equation (D) and integration applied, but the resulting expression is difficult to use.

The ratio of specific heats ($k = C_p/C_v$) for H_2 over the same temperature range varies only between 1.3 and 1.4. Using this parameter makes the resulting equations easier to use.

$$\text{Since } C_p = \frac{k}{k-1}\left(\frac{R}{J}\right)$$

where: R = gas constant $\left(\frac{\text{ft lb}}{\text{lb}\ ^\circ\text{R}}\right)$

$$h_i - h_e = \int_e^i \frac{k}{k-1}\left(\frac{R}{J}\right) dT$$

If an average of k between 1.3 and 1.4 is used. Then:

$$h_i - h_e = \frac{k}{k-1}\left(\frac{R}{J}\right)\int_e^i dT$$

$$\text{and: } h_i - h_e = -\frac{k}{k-1}\left(\frac{R}{J}\right)(T_e - T_i) \tag{E}$$

If equation (C) is solved for v_e (with $v_o \approx 0$) and $h_i - h_e$ from equation (E) is substituted:

$$v_e = \sqrt{2gJ\left[\nu\, q - \frac{k}{k-1}\left(\frac{R}{J}\right)(T_e - T_i)\right]} \tag{F}$$

Assuming the best case in which exhaust and ambient pressure are equal and pressure thrust equals zero:

$$F = \frac{\dot{W}}{g} v_e = \text{thrust (lb)}$$

Using equation (F) yields:

$$F = \frac{\dot{W}}{g}\sqrt{2\ g\ J\left[\nu\ q - \frac{k}{k-1}\left(\frac{R}{J}\right)(T_e - T_i)\right]} \qquad \text{(G)}$$

$$\text{or: } F = \frac{1}{g}\sqrt{2\,g\,J\,\dot{W}\left[\nu\,\dot{W}\,q - \dot{W}\left(\frac{k}{k-1}\right)\left(\frac{R}{J}\right)(T_e - T_i)\right]}$$

Recall that $\dot{W}\ q = Q$

$$\text{then: } F = \frac{1}{g}\sqrt{2\,g\,J\,\dot{W}\left[\nu Q - \dot{W}\left(\frac{k}{k-1}\right)\left(\frac{R}{J}\right)(T_e - T_i)\right]} \qquad \text{(H)}$$

For these constant values:

$$\begin{aligned} g &= 32.2 \text{ ft/sec}^2 \\ J &= 778.2 \text{ ft lb/BTU} \\ k &= \text{average value of } 1.35 \\ R &= 766.54\ \frac{\text{ft lb}}{\text{lb }^\circ\text{R}} \quad \text{(for hydrogen)} \end{aligned}$$

equation (H) becomes:

$$F = 6.94\sqrt{\dot{W}\,[\nu\,Q - 3.76\,\dot{W}\,(T_e - T_i]} \qquad \text{(I)}$$

In equation (I) Q, reactor thermal power, has units of BTU/sec. However, reactor power is usually represented in megawatts (MW). Employing the conversion factor 947 BTU/sec/MW in equation (I) yields:

$$F = 6.94\sqrt{\dot{W}\,[947\,\nu\,Q - 3.76\,\dot{W}\,(T_e - T_i)]}$$

(Equation 15, at end of Chapter 3)

where: F = thrust (lb)

$\dot{W}$ = hydrogen propellant flow rate (lb/sec)

Q = reactor *thermal* power (MW)

T = propellant bulk temperature (°R)

The parameter ν deserves further comment. It involves the efficiency of heat transfer between reactor structure and the propellant which, in turn, involves the study of heat transfer during forced convection. It involves the study of convective heat transfer under laminar and turbulent flow conditions. And, it involves the study of heat loss from the reactor core. These subjects are not within the scope of this handbook.

Appendix F

ACCURACY REQUIREMENTS FOR ORBITAL GUIDANCE

During the three phases of space flight—the injection phase, the midcourse trajectory, and the terminal phase—maintaining accuracy in guidance is a complex problem but one that is important to solve. The analysis given here provides a simple method of approximating the accuracy requirements for guidance during orbital missions. The orbit is assumed to be about a spherical, nonrotating earth in an environment free of atmospheric drag.

Consider first a satellite injected into orbit at a point r_{bo} from the center of the earth with a burnout speed v_{bo}. From Chapter 2, the general equation for an elliptical orbit is as follows:

$$v_{bo} = \sqrt{\frac{2\mu}{r_{bo}} - \frac{\mu}{a}}$$

or

$$a = \frac{\mu\, r_{bo}}{2\mu - v_{bo}^2\, r_{bo}}$$

Consider small errors (δ) in burnout radius δr_{bo} and burnout speed δv_{bo} and their effect δa upon the semimajor axis. Define δr_{bo} and δv_{bo} as small errors in burnout radius and burnout velocity.

$$r_{bo} \gg \delta r_{bo};\; v_{bo} \gg \delta v_{bo}$$

The semi-major axis of the ellipse defined by the satellite injected into the orbit, with a flight path angle of zero degrees, a burnout radius $r_{bo} + \delta r_{bo}$, and burnout velocity $v_{bo} + \delta v_{bo}$, is $a + \delta a$.

Then:

$$a + \delta a = \frac{\mu(r_{bo} + \delta r_{bo})}{2\mu - (v_{bo} + \delta v_{bo})^2\,(r_{bo} + \delta r_{bo})}$$

Since (δv_{bo}) is very small:

$$(v_{bo} + \delta v_{bo})^2 \approx v_{bo}^2 + 2v_{bo}\,\delta v_{bo}$$

$$\therefore a + \delta a = \frac{\mu(r_{bo} + \delta\, r_{bo})}{2\mu - v_{bo}^2\, r_{bo} - 2v_{bo}\, r_{bo}\, \delta v_{bo} - v_{bo}^2 \delta r_{bo}}$$

Subtracting:

$$\delta a = \frac{\mu r_{bo} + \mu \delta r_{bo}}{2\mu - v_{bo}^2 r_{bo} - 2v_{bo} r_{bo} \delta v_{bo} - v_{bo}^2 \delta r_{bo}} - \frac{\mu r_{bo}}{2\mu - v_{bo}^2 r_{bo}}$$

Making a common denominator of:

$$(2\mu - v_{bo}^2 r_{bo} - 2v_{bo} r_{bo} \delta v_{bo} - v_{bo}^2 \delta r_{bo})\ (2\mu - v_{bo}^2 r_{bo})$$

$$\delta a = \frac{2\mu^2 r_{bo} + 2\mu^2 \delta r_{bo} - \mu v_{bo}^2 r_{bo}^2 - \mu v_{bo}^2 \delta r_{bo} r_{bo}}{\text{Common denominator}}$$
$$- \frac{2\mu^2 r_{bo} + \mu v_{bo}^2 r_{bo}^2 + 2\mu v_{bo} r_{bo}^2 \delta v_{bo} + \mu v_{bo}^2 r_{bo} \delta r_{bo}}{\text{Common denominator}}$$

Combining terms in numerator:

$$\delta a = \frac{2\mu^2 \delta r_{bo} + 2\mu v_{bo} r_{bo}^2 \delta v_{bo}}{(2\mu - v_{bo}^2 r_{bo} - 2v_{bo} r_{bo} \delta v_{bo} - v_{bo}^2 \delta r_{bo})\ (2\mu - v^2_{bo} r_{bo})}$$

Since 2μ and $v_{bo}^2 r_{bo}$ are much greater than either

$2v_{bo} r_{bo} \delta v_{bo}$ or $v_{bo}^2 \delta r_{bo}$ then,

$$\delta a = \frac{2\mu^2 \delta r_{bo} + 2\mu v_{bo} r_{bo}^2 \delta v_{bo}}{(2\mu - v_{bo}^2 r_{bo})^2}$$

$$\frac{\delta a}{a} = \left[\frac{2\mu^2 \delta r_{bo} + 2\mu v_{bo} r_{bo}^2 \delta v_{bo}}{(2\mu - v_{bo}^2 r_{bo})^2}\right]\left[\frac{2\mu - v_{bo}^2 r_{bo}}{\mu r_{bo}}\right]$$
$$= \frac{2\mu \delta r_{bo} + 2v_{bo} r_{bo}^2 \delta v_{bo}}{(2\mu - v_{bo}^2 r_{bo})\ (r_{bo})}$$

Multiply top and bottom of right-hand side of above equation by μr_{bo}:

$$\frac{\delta a}{a} = \frac{\mu r_{bo}(2\mu \delta r_{bo})}{(2\mu - v_{bo}^2 r_{bo})\ (\mu r_{bo}^2)} + \frac{\mu r_{bo} (2v_{bo} r_{bo}^2 \delta v_{bo})}{(2\mu - v_{bo}^2 r_{bo})\ (\mu r_{bo}^2)}$$

$$\frac{\delta a}{a} = a\left[\frac{2}{r_{bo}^2} \delta r_{bo} + \frac{2v_{bo}}{\mu} \delta v_{bo}\right] \tag{1}$$

This equation expresses the change in semimajor axis of an elliptical orbit for small changes or errors in burnout radius and burnout speed.

Next, consider changes in orbital period δP for errors in burnout speed and burnout radius:

$$P = \frac{2\pi a^{3/2}}{\sqrt{\mu}}$$

and $a = \dfrac{\mu r}{2\mu - v^2 r}$, or for burnout conditions

$$a = \frac{\mu r_{bo}}{2\mu - v_{bo}^2 r_{bo}}$$

Therefore:

$$P = \frac{2\pi}{\sqrt{\mu}}\left(\frac{\mu\, r_{bo}}{2\mu - v_{bo}^2\, r_{bo}}\right)^{3/2}$$

Applying partial differentiation:

$$\frac{\partial P}{\partial v} = \frac{2\pi}{\sqrt{\mu}}\,(\mu r_{bo})^{3/2}\left(\frac{-3}{2}\right)\left(2\mu - v_{bo}^2\, r_{bo}\right)^{-5/2}\left(-2v_{bo}\, r_{bo}\right)$$

$$\text{Since } P = \frac{2\pi}{\sqrt{\mu}}\,\frac{(\mu r_{bo})^{3/2}}{(2\mu - v_{bo}^2 r_{bo})^{3/2}}$$

$$\frac{\partial P}{\partial v} = \frac{3P\, r_{bo}\, v_{bo}}{2\mu - v_{bo}^2\, r_{bo}}$$

Multiply top and bottom of right side by μ:

$$\frac{\partial P}{\partial v} = 3P\left(\frac{\mu r_{bo}}{2\mu - v_{bo}^2\, r_{bo}}\right)\left(\frac{v_{bo}}{\mu}\right)$$

$$\frac{\partial P}{\partial v} = \frac{3Pa\, v_{bo}}{\mu}$$

$$\therefore\ \delta P \approx \frac{3a\ P\ v_{bo}}{\mu}\ \delta\ v_{bo} \qquad (2)$$

Derivation of Equation 3 is similar to that of Equation 2.

$$\delta P = \frac{3aP}{r_{bo}^2}\ \delta r_{bo} \qquad (3)$$

For a nearly circular orbit:

$$a \approx r_{bo} \text{ and } v_{bo}^2 \approx \frac{\mu}{a}$$

Equation 1 can then be simplified to:

$$\delta a = 2\delta r_{bo} + \frac{2a}{v_{bo}}\ \delta v_{bo} \qquad (4)$$

The effect of a small error in burnout flight path angle, $\delta\phi_{bo}$, is somewhat complicated for an elliptical orbit. However, if a circular orbit were desired, then the eccentricity of the orbit caused by the velocity vector not being exactly horizontal (i.e.,

$$\epsilon \approx \delta\phi_{bo} \qquad (5)$$

where $\delta\phi_{bo}$ is expressed in radians.

Derivation of Equation 5:

The relation between the eccentricity of an orbit and its total specific mechanical energy and angular momentum was given by equation 6, Chapter 2 as:

$$\epsilon = \sqrt{1 + \frac{2EH^2}{\mu^2}}$$

Substituting

$$E = \frac{v^2}{2} - \frac{\mu}{r} \text{ and } H = vr \cos \phi$$

into the first equation:

$$\epsilon = \left[1 + \frac{2}{\mu^2}\left(\frac{v^2}{2} - \frac{\mu}{r}\right) v^2r^2 \cos^2 \phi\right]^{½}$$

$$= \left[1 + \frac{v^4r^2}{\mu^2} (1 - \frac{2\mu}{v^2r}) \cos^2 \phi \right]^{½}$$

Making use of the definition from Appendix C:

$$Q_{bo} = \frac{v_{bo}^2\, r_{bo}}{\mu}$$

$$\text{then } \epsilon = \left[1 + Q_{bo}^2 \ (1 - \frac{2}{Q_{bo}}) \cos^2 \phi_{bo}\right]^{½}$$

Specializing to the case of a circular orbit, where $Q_{bo} = 1$

$\epsilon = [1 - \cos^2 \phi_{bo}]^{½} = [\sin^2 \phi_{bo}]^{½} \sin \phi_{bo}$

For a circular orbit, $\epsilon = \sin \delta \phi_{bo}$

For small errors in flight path angle, ($\delta \phi_{bo} = \sin \delta \phi_{bo}$)

$$\epsilon = \delta \phi_{bo} \tag{5}$$

The next step will be to apply the equations for small guidance errors to representative orbits. Since circular orbits have important practical application today, what representative accuracies are required?

Problem 1. An earth satellite is to be injected into a circular orbit at 300 NM above the earth. Assume that guidance errors cause the vehicle to be injected into orbit at 301 NM altitude with a speed at burnout 1 ft/sec higher than desired.

Find: Actual height at apogee of the resulting ellipse.

Circular speed at 300 NM = 24,900 ft/sec

Using Equation 4:

$$\delta a = 2\, \delta r_{bo} + 2a \frac{\delta v_{bo}}{v_{bo}}$$

$$\delta r_{bo} = 301 \text{ NM} - 300 \text{ NM} = 1 \text{ NM}$$

$$\delta v_{bo} = 1 \text{ ft/sec}$$

$$a = r_e + 300 \text{ NM} = 3440 \text{ NM} + 300 \text{ NM} = 3740 \text{ NM}$$

$$\delta a = (2)\ (1 \text{ NM}) + \frac{(2)\ (3740 \text{ NM})\ (1 \text{ ft/sec})}{24{,}900 \text{ ft/sec}}$$

$$\delta a = 2 \text{ NM} + .300 \text{ NM} = 2.3 \text{ NM}$$

The distance from perigee to apogee equals twice the semimajor axis of the elliptical orbit (Figure 1).

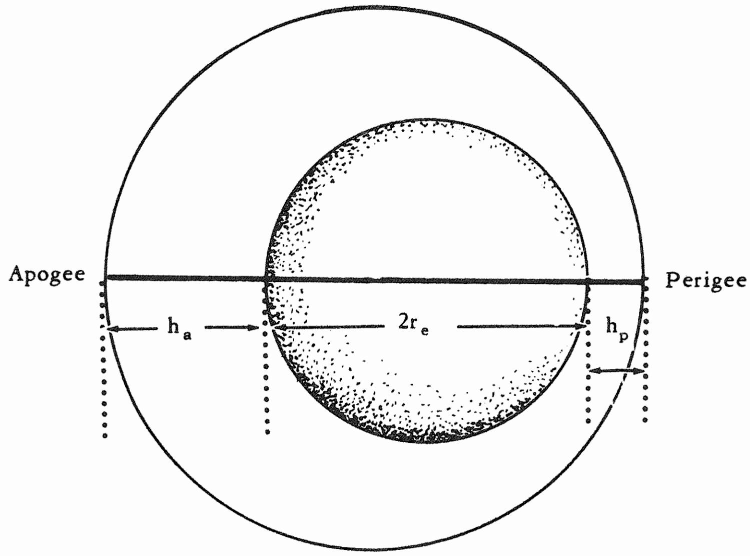


Figure 1. Ellipse formed by orbiting earth satellite.

Therefore,

$$h_a + 2r_e + h_p = 2(a + \delta a)$$

$$h_a + 6880\text{ NM} + 301\text{ NM} = 2(3740 + 2.3\text{ NM})$$

$$h_a = 7484.6\text{ NM} - 6880\text{ NM} - 301\text{ NM} = 303.6\text{ NM}$$

In the above problem, assume that injection occurred at the exact speed and altitude desired (i.e., $\delta v_{bo} = \delta r_{bo} = 0$), but that the burnout angle was 0.1°. Find the eccentricity of the resulting orbit, and the height of apogee.

$$\epsilon = \delta\phi_{bo} = \frac{0.1°}{57.3°/\text{rad}} = 0.001745$$

$$c = \epsilon\alpha = 0.001745 \times 3740\text{ NM} = 6.5\text{ NM}$$

$$r_a = a + c = 3740 + 6.5 = 3746.5\text{ NM}$$

$$h_a = r_a - r_e = 3746.5\text{ NM} - 3440\text{ NM} = 306.5\text{ NM}$$

Problem 2: Consider next a Hohmann transfer from a 300 NM orbit to 19,360 NM altitude. Find altitude at apogee for a guidance error at perigee of the transfer ellipse of:

(1) $\delta v_{bo} = 1$ ft/sec high, no altitude or burnout angle error

(2) $\delta r_{bo} = 6080$ ft high, no errors in speed or angle.

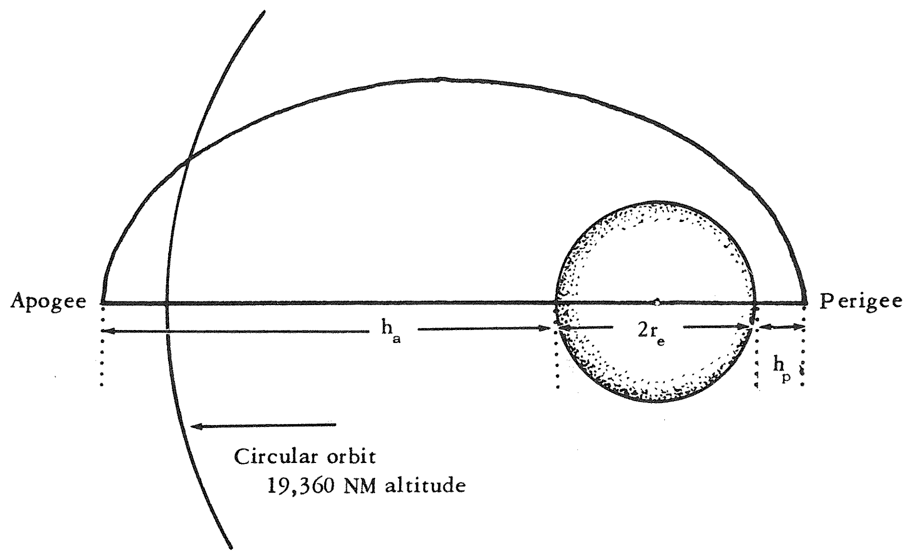


Figure 2. Ellipse formed in the Hohmann transfer.

Using Equation 1:

$$\frac{\delta a}{a} = a \left[\underset{\text{Altitude error}}{\frac{2\ \delta r_{bo}}{r_{bo}^2}} + \underset{\text{Velocity error}}{\frac{2v_{bo}\ \delta v_{bo}}{\mu}} \right]$$

(1) For speed error:

$$\frac{\delta a}{a} = a\ \frac{2v_{bo}\delta v_{bo}}{\mu}$$

$$2a = 19{,}360 \text{ NM} + 6880 \text{ NM} + 300 \text{ NM}$$

$$a = 13{,}270 \text{ NM}, \ v_{bo} = 32{,}600 \text{ ft/sec}$$

$$\delta a = \frac{(13{,}270 \text{ NM} \times 6080 \text{ ft/NM})^2 (2)\ (32{,}600 \text{ ft/sec})\ (1 \text{ ft/sec})}{14.08 \times 10^{15} \text{ ft}^3/\text{sec}^2}$$

$$= 30{,}200 \text{ ft} \approx 5 \text{ NM}$$

The distance from perigee to apogee equals twice the semimajor axis of the ellipse attained. See Figure 2 for the major axis.

Therefore,

$$h_a + 2r_e + h_p = 2(a + \delta a)$$

$$h_a + 300 \text{ NM} + 6880 \text{ NM} = 2(13{,}270 \text{ NM} + 5 \text{ NM}) = 26{,}550 \text{ NM}$$

$$h_a = 19{,}370$$ *Answer*

(2) For altitude error:

$$\frac{\delta a}{a} = a\left(\frac{2\delta r_{bo}}{r_{bo}^2}\right)$$

$$\delta a = \frac{(13{,}270\text{ NM})^2(2)\ 1\text{ NM}}{(3740\text{ NM})^2} = 25.2\text{ NM} \approx 25\text{ NM}$$

$$h_a + h_p + 2r_e = 2(a + \delta a) = 2(13{,}270\text{ NM} + 25\text{ NM})$$

$$h_a + 301\text{ NM} + 6880\text{ NM} = 26{,}590\text{ NM}$$

$$h_a = 19{,}409\text{ NM}$$

Answer

GLOSSARY OF SYMBOLS

1.	a	semi-major axis of ellipse; average linear acceleration
2.	b	semi-minor axis of ellipse
3.	bo	subscript for burnout conditions
4.	c	distance between focus and center of ellipse
5.	e	base of natural logarithm, = 2.718
6.	g	local acceleration due to gravity, = 32.2 ft/sec^2 at the surface of the earth
7.	h	altitude, height above surface of earth
8.	h_a	altitude of apogee
9.	h_p	altitude of perigee
10.	i	angle of inclination
11.	p	electric power
12.	r	radius length; mixture ratio
13.	r_a	radius to apogee
14.	r_e	radius of earth, = 3440 NM = 20.9×10^6 ft
15.	r_p	radius to perigee
16.	s	linear displacement
17.	t	time in sec
18.	u	eccentric anomaly
19.	v	linear speed, velocity magnitude
20.	v_e	nozzle exit velocity
21.	Δv	increment or change of speed
22.	w	work in ft-lb force
23.	A	area
24.	A_e	nozzle exit area
25.	C_D	coefficient of drag
26.	C_L	coefficient of lift
27.	D	atmospheric drag
28.	E	specific mechanical energy in ft^2/sec^2

29.	F	focus of ellipse; force in lbs; thrust in lbs
30.	G	Universal Gravitational Constant, $= 10.69 \times 10^{-10}$ ft^3/lb mass-sec^2
31.	H	specific angular momentum in ft^2/sec
32.	I_{sp}	specific impulse in sec
33.	I_t	total impulse in lb-secs
34.	KE	kinetic energy
35.	L	aerodynamic lift; latitude
36.	M	mass in slugs
37.	$\dot{M}$	mass flow rate
38.	NM	nautical miles, = 6080 ft
39.	P	period of revolution; pressure
40.	PE	potential energy
41.	Q	ballistic trajectory parameter; reactor thermal power
42.	W	weight in pounds force
43.	$\dot{W}$	weight flow rate
44.	α	average angular acceleration
45.	δ	very small change or error
46.	ε	eccentricity; expansion ratio
47.	η	electrical efficiency
48.	θ	angular displacement
49.	λ	longitude
50.	μ	gravitational parameter, $= 14.08 \times 10^{15}$ ft^3/sec^2 for earth
51.	ν	true anomaly; heat transfer efficiency
52.	π	conversion constant, = 3.1416; π radians = 180°
53.	ρ	atmospheric density, slugs/ft^3
54.	ϕ	flight path angle, elevation angle of velocity vector
55.	ψ	free flight range angle
56.	ω	argument of perigee; average angular speed
57.	Δ	increment of
58.	Ψ	thrust to weight ratio

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